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A Generalized Cramer's Rule for Tri- Component Interval -valued Neutrosophic Linear Systems

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Abstract: This paper introduces a novel method for solving tri-component interval-valued Neutrosophic linear systems. Building upon fundamental concepts of Neutrosophic sets, including tri-component interval numbers and their algebraic operations, we first derive a generalized matrix representation for systems with *n* linear equations with *m* unknowns in this uncertain environment. The core contribution of this work is the development of a generalized Cramer's rule tailored for these Neutrosophic systems, providing an analytical framework for obtaining solutions under conditions of incompleteness, inconsistency, and indeterminacy. The efficacy and robustness of the proposed method are demonstrated through compassing numerical examples, encompassing binary and a generalized system cases. These examples illustrate all possible types of solution: unique solution, no solution, and infinitely many solutions. This research focuses on the theoretical and analytical aspects of solving Neutrosophic systems, relying on Cramer's rule to find abstract mathematical solutions.

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قاعدة كرامر معممة للأنظمة الخطية النيوتروسوفية ثلاثية المكونات ذات القيم الفاصلة

خديجة بشير صولة

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المستخلص: يقدم هذا البحث طريقة جديدة لحل أنظمة المعادلات الخطية النيوتروسوفية ثلاثية المكونات ذات القيم الفاصلة. بالاعتماد على المفاهيم الأساسية للمجموعات النيوتروسوفية، والتي تشمل الأعداد الفاصلة ثلاثية المكونات وعملياتها الجبرية، نشتق أولاً تمثيلًا مصفوفيًا معممًا للأنظمة المكونة من عدد n من المعادلات الخطية وعدد m من المجاهيل في هذه البيئة غير المؤكدة. يتمثل الإسهام الجوهري في هذا العمل في تطوير قاعدة كرامر معممة خصيصًا لهذه الأنظمة النيوتروسوفية، مما يوفر إطارًا تحليليًا للحصول على الحلول في ظل ظروف النقص والتناقض وعدم التحديد. يتم إثبات فعالية ومتانة الطريقة المقترحة من خلال أمثلة عددية شاملة، تتضمن حالات أنظمة ثنائية وأخرى معممة. توضح هذه الأمثلة جميع الأنواع الممكنة للحل: حل وحيد، لا يوجد حل، وعدد لا نهائي من الحلول. يركز هذا البحث على الجوانب النظرية والتحليلية لحل الأنظمة النيوتروسوفية، معتمدًا على قاعدة كرامر لإيجاد حلول رياضية مجردة.

الكلمات المفتاحية
نيوتروسوفي
أنظمة خطية
المحددات
قاعدة كرامر
ليبيا

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1. Introduction

The concept of neutrosophic logic and its associated mathematical foundations has witnessed significant development since its introduction by Smarandache [20,21,23], providing a unified framework that generalizes classical logic, fuzzy logic and intuitionistic logic, allowing the representation of uncertainty vagueness, and indeterminacy in complex systems [1,22].

With the evolution of neutrosophic set structures, research has extended to matrices and algebraic operations, enabling applications in optimization, decision-making, and mathematical modeling [6,7,11,26]. Classical linear algebra techniques have also been extended to neutrosophic systems, where determinants and Cramer's rule have been employed to solve neutrosophic linear equations, accommodating imprecise coefficients represented as intervals [1,7,13].

These developments have contributed to practical applications, such as traffic flow modeling, system optimization, and decision support [15,26,27,29]. Moreover, the algebraic structures associated with neutrosophic numbers, including neutrosophic rings and matrices, have been rigorously analyzed, facilitating the development of linear programming solutions under uncertainty and extending classical computational methods to tri- component interval representations [6,16,30].

These efforts demonstrate the flexibility of neutrosophic mathematics in linking abstract algebraic concepts with practical decision-making applications [5,24,30]. Historically, neutrosophic linear algebra techniques are based on the classical matrix and determinant theories developed by [3,4,8,9,10,12,14,17,18,19,25,28].

These foundations have been further extended to handle imprecise data and intervals, providing robust frameworks for solving linear systems with inexact coefficients [12,19].

Recent contributions have improved computational operations on neutrosophic matrices and solution techniques, enhancing the accuracy and efficiency of solutions for complex systems [4,10,23, 28]. This study aims to bridge the gap in the literature regarding tri- component interval-valued neutrosophic linear systems by proving a generalized matrix form and an extended Cramer's rule. The validity of the proposed approach is verified through comprehensive numerical examples, offering a direct and effective analytical tool to address all possible solution scenarios [1,7,16,28].

This research aims to bridge the theoretical and practical gap in the study of tri- component interval-valued neutrosophic linear systems by deriving a generalized $n \times n$ matrix formulation that rigorously represents such systems within the neutrosophic framework. A major contribution of this work is the development of a generalized Cramer's rule that enables the explicit analytical solution of these systems, covering all possible cases: unique solution, infinitely many solutions, and no solution. The proposed approach integrates theoretical formulation with comprehensive numerical verification through illustrative examples for both 2×2 and general $n \times n$ systems, thereby demonstrating the accuracy, robustness, and elegance of the method as a natural extension of classical linear algebra theory in a more realistic and complex setting. The outcomes of this study are expected to enrich the field of neutrosophic algebra and provide a powerful analytical tool for researchers in optimization and decision-making under uncertainty.

2. Preliminaries

This section establishes the fundamental mathematical concepts and notations required for the rigorous analysis of tri-component interval-valued neutrosophic linear systems. Formal definitions are presented to ensure precision and logical consistency. And also introduces the notions of truth (T), indeterminacy (I), and falsity (F) interval, elaborating on their properties, interrelations, and the rules governing arithmetic and algebraic operations within neutrosophic structures.

Definition 2.1 [20,22]: (Neutrosophic Set): Let X be a universe. A neutrosophic set A in X is characterized by three membership functions:

Truth-membership function $T_A(x)$,

Indeterminacy-membership function $I_A(x)$ and

Falsity-membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$ and $0 \leq sup\ T_A(x) + supI_A(x) + supF_A(x) \leq 3\ \forall\ x\ \in X$.

Definition 2.2 [22,27]: (Tri-Component Interval Neutrosophic Number):

An ordered triple of intervals is represented as: $\tilde{a}$ = $([T^L, T^U], [I^L, I^U], [F^L, F^U])$

Where:

$[T^L, T^U] \subseteq [0, 1]$ is the truth-membership interval, $[I^L, I^U] \subseteq [0, 1]$ is

the indeterminacy-membership interval , $[F^L, F^U] \subseteq [0, 1]$ is the falsity-membership interval. And satisfying:

$0 \leq T^L \leq T^U \leq 1,\ \ 0 \leq I^L \leq I^U \leq 1,\ \ 0 \leq F^L \leq F^U \leq 1,$

$0 \leq T^U + I^U + F^U \leq 3,$

$\max(T^L, I^L, F^L) > 0$

Not that,$[T^L, T^U]$: Upper and lower intervals of the truth component, $[I^L, I^U]$: Upper and lower intervals of the indeterminacy component, $[F^L, F^U]$: Upper and lower intervals of the falsity component.

Example 2.2 : $\tilde{a} = ([0.6,0.8], [0.1,0.3], [0.0,0.5])$

Definition 2.3 [5,22]: (Neutrosophic Algebraic Operations)

For two Tri-Component Interval neutrosophic Numbers

$\tilde{a} = ([T_1^L,\ T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])$ and $\tilde{b} = ([T_2^L,\ T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])$. Then:

Addition:

$\tilde{a} \oplus \tilde{b} = ([T_1^L + T_2^L - T_1^L T_2^L,\ T_1^U + T_2^U - T_1^U T_2^U], [(I_1^L I_2^L),\ (I_1^U I_2^U)], [(F_1^L F_2^L),\ (F_1^U F_2^U)]).$

Multiplication:

$\tilde{a} \odot \tilde{b} = ([T_1^L T_2^L, T_1^U T_2^U], [(I_1^L + I_2^L) - (I_1^L I_2^L), (I_1^U + I_2^U) - (I_1^U I_2^U)], [(F_1^L + F_2^L) - (F_1^L F_2^L), (F_1^U + F_2^U) - (F_1^U F_2^U)],$

Multiplication by scaler $\boldsymbol{\lambda} \in [\mathbf{0}, \mathbf{1}]$:

$\lambda\,.\,\tilde{a} = \lambda\,.\,([T_1^L, T_1^U], I_1^L, I_1^U], [F_1^L, F_1^U])$

$= \left([1 - (1 - T_1^L)^\lambda, 1 - (1 - T_1^U)^\lambda\right], [(I_1^L)^\lambda, (I_1^U)^\lambda], [(F_1^L)^\lambda, (F_1^U)^\lambda])$

Subtraction:

$$\tilde{a} \ominus \tilde{b} = (\left[\frac{T_1^L - T_2^U}{1 - T_2^U}, \frac{T_1^U - T_2^L}{1 - T_2^L}\right], \left[\frac{I_1^L}{I_2^U}, \frac{I_1^U}{I_2^L}\right], [\frac{F_1^L}{F_2^U}, \frac{F_1^U}{F_2^L}])$$

Division:

$$\tilde{a} \oslash \tilde{b} = \left(\left[\frac{T_1^L}{T_2^U}, \frac{T_1^U}{T_2^L}\right], \left[\frac{I_1^L - I_2^U}{1 - I_2^U}, \frac{I_1^U - I_2^L}{1 - I_2^L}\right], \left[\frac{F_1^L - F_2^U}{1 - F_2^U}, \frac{F_1^U - F_2^L}{1 - F_2^L}\right]\right); [T_2^U, T_2^L] \notin 0 \wedge 1, [F_2^U, F_2^L] \notin 1$$

Remarks 2.3 [22]:

These operations preserve the closure property for interval Neutrosophic numbers.

Example 2.3: Let $\tilde{a} = ([0.6, 0.8], [0.1,0.3][0.0, 0.1])$ and

$\tilde{b} = ([0.5,0.7]\ [0.2\ , 0.4], [0.1, 0.2])$ Then:

$\tilde{a} \oplus \tilde{b} = ([\ 0.8, 0.94], [0.02, 0.12], [0.0, 0{,}02])$

$\tilde{a} \otimes \tilde{b} = ([\ 0.3, 0.56], [0.28, 0.58], [0.1, 0{,}28])$,

$\tilde{a} \ominus \tilde{b} \ = ([-\ 0.33, 0.6], [0.25, 1.5], [0.0, 1])$,

$\tilde{a} \oslash \tilde{b} = ([\ 0.857, 1.54], [-0.5, 0.125], [-0.25, 0.0])$,

$\lambda \ \ \tilde{a} = \lambda\ ([0.6, 0.8], [0.1, 0.3][0.0, 0.1]);\ \lambda = 0.2\ \in [0, 1]$

$= ([1-(1-0.6)^{\mathbf{0.2}}, 1-(1-0.8)^{\mathbf{0.2}}], [(0.1)^{\mathbf{0.2}}, (0.3)^{\mathbf{0.2}}], [(0.0)^{\mathbf{0.2}}, (0.1)^{\mathbf{0.2}}])$

= ([0.167,0.275], [0.631,0.786], [0.0,0.631])

Remarks 2.3 [1,7]:

1. All Neutrosophic component are expressed as ordered triplets of intervals.

2. Arithmetic operations are performed component wise over each of *T, I, F* intervals using interval arithmetic.

3. The result of any operation remains the format of single interval neutrosophic number

Definition 2.4 [2,23]: A matrix $A = \left(\tilde{a}_{ij}\right)_{m\times n}$ is called a tri- component interval neutrosophic matrix if every entry $\tilde{a}_{ij}$; (i =1, 2,…,m; j = 1, 2, …,n) is a tri- component interval neutrosophic number defined as:

$(\tilde{a}_{ij}) = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$

Where:

$[T_{ij}^L, T_{ij}^U] \subseteq [0, 1]$: represents the truth degrees, $[I_{ij}^L, I_{ij}^U] \subseteq [0, 1]$: represents the indeterminacy degrees , $[F_{ij}^L, F_{ij}^U] \subseteq [0, 1]$: represents the falsity degrees.

Definition 2.5 [1,7,9]: (Neutrosophic Determinant for a $\mathbf{2} \times \mathbf{2}$ matrix).

Let $A = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$ be a 2×2 Neutrosophic matrix. Its neutrosophic determinant is defined and computed as: $\det(A) = \tilde{a}_{11}\ \tilde{a}_{22} + (-\ \tilde{a}_{12})\tilde{a}_{21})$.

where all operations are performed in definition 2.3.

Definition 2.6 [1,7,9]: (Neutrosophic Determinant for an n × n matrix).

For a square neutrosophic matrix A of order $n \times n$ with elements $[T_{ij}, I_{ij}, F_{ij}]$, the neutrosophic determinant is calculated through the relation:

$\det(A) = (\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)}, \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} I_{i,\sigma(i)}, \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} F_{i,\sigma(i)})$

Where S_n represents the set of all permutations of n, and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

Example 2.6:

Let neutrosophic matrix A =

$$\begin{pmatrix} [0.8,0.1,0.1] & [0.6,0.2,0.2] & [0.7,0.1,0.2] \\ [0.8,0.3,0.2] & [0.9,0.1,0.0] & [0.4,0.2,0.4] \\ [0.6,0.2,0.2] & [0.3,0.4,0.3] & [0.5,0.3,0.2] \end{pmatrix}$$

From definition 2.3 and Calculation per Component Using Sarrus' Rule [1, 2]:

Truth Component (T): $\det(T) = \begin{vmatrix} 0.8 & 0.6 & 0.7 \\ 0.8 & 0.9 & 0.4 \\ 0.6 & 0.3 & 0.5 \end{vmatrix}$

= 0.8(0.9. 0.5-0.4. 0.3) - 0.6(0.8. 0. 5-0.4. 0.6) + 0.7(0.8. 0.3-0.9. 0.6) = - 0.042

Indeterminacy Component (I): $\det(I) = \begin{vmatrix} 0.1 & 0.2 & 0.1 \\ 0.3 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.3 \end{vmatrix}$

= 0.1(0.1. 0.3-0.2 . 0.4) - 0.2(0.3. 0.3-0.2. 0.2) + 0.1(0.3. 0.4-0.1. 0.2) = 0.067

Falsity Component (F): $\det(F) = \begin{vmatrix} 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.2 \end{vmatrix}$

= 0.1(0.2. 0.2-0.4. 0.3) - 0.2(0.2. 0.2-0.4. 0.2) + 0.2(0.2. 0.3-0.2. 0.2) = 0.004

the neutrosophic determinant: det (A) = [- 0.042, 0.067, 0.004]

Definition 2.8 [16]: The systems of neutrosophic linear equations in which the number of equations equals m and the number of n variables in them equals $n$are given in the following general form:

$$\begin{cases} Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n = Nb_1 \\ Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n = Nb_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Na_{n1}x_1 + Na_{n2}x_2 + \cdots + Na_{mn}x_n = Nb_m \end{cases} \quad \text{.......................................} (1)$$

In the following matrix form:

$NA. X = NB$ .... (2)

Where:

$$NA = \begin{pmatrix} Na_{11} & Na_{12} & \cdots & Na_{1n} \\ Na_{21} & Na_{22} & \cdots & Na_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Na_{n1} & Na_{n2} & \cdots & Na_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}, NB = \begin{pmatrix} Nb_1 \\ Nb_2 \\ \vdots \\ \vdots \\ Nb_n \end{pmatrix}.$$

where Nb_i and Na_{ij}, neutrosophic numbers, written in one of the forms. next: $Naij$=aij+εij and $Nb_i = b_i + \mu_i$ where $\varepsilon_{ij} \in [\lambda_{1ij}, \lambda_{1ij}]$ or otherwise.

3. Tri-Component Interval-Valued Neutrosophic Linear Systems

In the pursuit of modeling systems fraught with inherent uncertainty, inconsistency, and indeterminacy, classical mathematical frameworks often reveal their limitations. The need to capture the nuanced nature of real-world phenomena-where truth, falsity, and indeterminacy coexist independently and interactively-has led to the emergence of neurotrophic logic as a transformative paradigm. This section delves into the foundational conceptualization of tri-components interval-valued neutrosophic linear systems, a sophisticated structure designed to embrace and operationalize this complexity.

At its core, this framework represents a departure from reductionist approaches that compress uncertainty into singular values or binary classifications. Instead, it acknowledges that uncertainty is not monolithic but multifaceted. Each component-truth (T), indeterminacy (I), and falsity (F)-is accorded its own ontological status, represented not as precise points but as intervals. This interval-valued characterization reflects the inherent vagueness and fluidity of real-world information, where boundaries are often blurred and knowledge is imperfect. The general form of such a system is conceived as a network of linear equations where coefficients and variables are not numbers but triads of intervals. This structure captures the simultaneous presence of partial truth, partial falsehood, and partial ignorance in every element of the system. The matrix representation of this system extends classical linear algebra into a richer algebraic space, where matrices and vectors are no longer mere arrays of numbers but organized collections of these triadic interval structures.

To navigate this complexity, the framework employs a component-wise decomposition, philosophical commitment to analyzing each dimension of truth, indeterminacy, and falsity both independently and synergistically. This decomposition is not merely a technical tool but an epistemological stance: it asserts that understanding the whole requires examining its constituent dimensions without forcing premature synthesis. Complementing this is the tri-components partition representation, which organizes the system into coherent subsystems along these ontological divisions, while preserving the integrity of each component. Central to solving these systems is the novel concept of tri-components determinants. Unlike classical determinants, which scalarize matrix properties, these determinants retain the triadic structure, embodying the system's inherent uncertainty even in its most condensed representation. They serve as the gateway to defining a neutrosophic solution vector-a solution that is itself a triad of intervals, offering not a single answer but a spectrum of possible truths, indeterminacies, and falsehoods that satisfy the system under conditions of uncertainty. The tri-components neutrosophic structure is, therefore, more than a mathematical model; it is a philosophical framework for humility in the face of complexity.

It rejects the illusion of precision where none exists and offers instead a tool for honest engagement with the world's inherent ambiguities. This section lays the groundwork for this engagement, defining the structures that will later enable the extension of Cramer's rule into this expansive domain of neutrosophic logic.

Definition 3.1: A neutrosophic interval-valued linear system is a system of m equations with n variables where coefficients are a tri-component interval neutrosophic numbers. It is given by the form:

$$\left\{\begin{array}{c} ([T_{11}^L,\ T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U])x_1 \oplus ([T_{12}^L,\ T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U])x_2 \oplus \dots \oplus \\ ([T_{1n}^L,\ T_{1n}^U],[I_{1n}^L,I_{1n}^U],[F_{1n}^L,F_{1n}^U])x_n = \left([T_{b_1}^L,\ T_{b_1}^U],[I_{b_1}^L,I_{b_1}^U],[F_{b_1}^L,F_{b_1}^U]\right) \\ ([T_{21}^L,\ T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U])x_1 \oplus ([T_{22}^L,\ T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U])x_2 \oplus \dots \oplus \\ ([T_{2n}^L,\ T_{2n}^U],[I_{2n}^L,I_{2n}^U],[F_{2n}^L,F_{2n}^U])x_n = \left([T_{b_2}^L,\ T_{b_2}^U],[I_{b_2}^L,I_{b_2}^U],[F_{b_2}^L,F_{b_2}^U]\right) \\ \vdots \qquad\qquad \vdots \qquad \ddots \qquad \vdots \\ ([T_{n1}^L,\ T_{n1}^U],[I_{n1}^L,I_{n1}^U],[F_{n1}^L,F_{n1}^U])x_1 \oplus ([T_{n2}^L,\ T_{n2}^U],[I_{n2}^L,I_{n2}^U],[F_{n2}^L,F_{n2}^U])x_2 \oplus \dots \oplus \\ ([T_{mn}^L,\ T_{mn}^U],[I_{mn}^L,I_{mn}^U],[F_{mn}^L,F_{mn}^U])x_n = \left([T_{b_m}^L,\ T_{b_m}^U],[I_{b_m}^L,I_{b_m}^U],[F_{b_m}^L,F_{b_m}^U]\right) \end{array}\right. \quad \dots\dots\dots (3)$$

Where $\oplus$ and . denoted the neutrosophic multiplication and Addition, respectively. an be represented in matrix form as:

$$A_N\, X_N = B_N \quad \dots\dots\dots (4)$$

Where:

A_N: An $n \times m$ matrix of tri-component interval neutrosophic coefficients:

$$\begin{pmatrix} ([T_{11}^L,\ T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U]) & ([T_{12}^L,\ T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U]) & \cdots & ([T_{1n}^L,\ T_{1n}^U],[I_{1n}^L,I_{1n}^U],[F_{1n}^L,F_{1n}^U]) \\ ([T_{21}^L,\ T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U]) & ([T_{22}^L,\ T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U]) & \cdots & ([T_{2n}^L,\ T_{2n}^U],[I_{2n}^L,I_{2n}^U],[F_{2n}^L,F_{2n}^U]) \\ \vdots & \vdots & \ddots & \vdots \\ ([T_{n1}^L,\ T_{n1}^U],[I_{n1}^L,I_{n1}^U],[F_{n1}^L,F_{n1}^U]) & ([T_{n2}^L,\ T_{n2}^U],[I_{n2}^L,I_{n2}^U],[F_{n2}^L,F_{n2}^U]) & & ([T_{mn}^L,\ T_{mn}^U],[I_{mn}^L,I_{mn}^U],[F_{mn}^L,F_{mn}^U]) \end{pmatrix}_{m\times n}$$

$$X_N = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}, \quad B_N = \begin{pmatrix} \left([T_{b_1}^L,\ T_{b_1}^U],[I_{b_1}^L,I_{b_1}^U],[F_{b_1}^L,F_{b_1}^U]\right) \\ \left([T_{b_2}^L,\ T_{b_2}^U],[I_{b_2}^L,I_{b_2}^U],[F_{b_2}^L,F_{b_2}^U]\right) \\ \vdots \\ \left([T_{b_m}^L,\ T_{b_m}^U],[I_{b_m}^L,I_{b_m}^U],[F_{b_m}^L,F_{b_m}^U]\right) \end{pmatrix}_{m\times 1}$$

X_N: Solution neutrosophic vector: $(x_j)_{n\times 1}$, B_N: Neutrosophic constant vector:

$([T^L_{b_i}, \ T^U_{b_i}], [I^L_{b_i}, I^U_{b_i}], [F^L_{b_i}, F^U_{b_i}])_{m\times 1}$ $\forall\, i \in \{1, 2, \dots, m\}$.

If $B_N = 0$ then the systems (4) called a homogeneous systems and otherwise, it is called a non-homogeneous systems.

Definition 3.2: A General tri-components interval-valued neutrosophic linear system is defined as follows:

$\sum_{j=1}^{n}(\tilde{a}_{ij} \cdot x_j) = \tilde{b}_i \quad i = 1, 2, \dots, m$ ………………………………………………………………………… *(5)*

Where:

$\tilde{a}_{ij} = \left([T^L_{ij}, \ T^U_{ij}], [I^L_{ij}, I^U_{ij}], [F^L_{ij}, F^U_{ij}]\right)$:

Tri-components interval-valued neutrosophic coefficient, x_j: unknown Neutrosophic variables,

$\tilde{b}_i = \left([T^L_{b_i}, \ T^U_{b_i}], [I^L_{b_i}, I^U_{b_i}], [F^L_{b_i}, F^U_{b_i}]\right)$: Neutrosophic right-hand side.

The system (5) with three- component coefficients (with upper and lower bounds) is represented by general form as:

$\sum_{j=1}^{n}([T^L_{ij}, \ T^U_{ij}], [I^L_{ij}, I^U_{ij}], [F^L_{ij}, F^U_{ij}]) \cdot x_j = \left([T^L_{b_i}, \ T^U_{b_i}], [I^L_{b_i}, I^U_{b_i}], [F^L_{b_i}, F^U_{b_i}]\right)$ ……………………………………………… *(6)*

Where:

$[T^L_{ij}, T^U_{ij}]$: Upper and lower intervals of the truth component of the coefficient,

$[I^L_{ij}, I^U_{ij}$: Upper and lower intervals of the indeterminacy component of the coefficient,

$[F^L_{ij}, F^U_{ij}]$: Upper and lower intervals of the falsity component of the coefficient.

with Conditions:

- Normalization Condition:

  $0 \le T^U_{ij} + I^U_{ij} + F^U_{ij} \le 3 \ \forall\, i, j$

- Non-contradiction Principle:

  $T^U_{ij} + F^U_{ij} \le 1$ and $I^L_{ij} \le \ \max(|\, T^U_{ij} - \ T^T_{ij} \,|, |\, F^U_{ij} - \ F^T_{ij} \,|) \ \ \forall\, i, j$

Components Independence: T, I, F (completely independent).

Definition 3.3: (Component-Wise Decomposition)

The system decomposes into three independent crisp interval-linear subsystems:

- Truth-Component system: $\sum_{j=1}^{n}[T^L_{ij}, \ T^U_{ij}] \cdot x_j \ = [T^L_i, \ T^U_i] \ \forall i$
- Indeterminacy -Component system:

$$\sum_{j=1}^{n}[I^L_{ij}, I^U_{ij}] \cdot x_j \ = [I^L_i, \qquad I^U_i] \ \forall i$$

- Falsity-Component system: $\sum_{j=1}^{n}[F^L_{ij}, F^U_{ij}] \cdot x_j \ = [F^L_i, \ F^U_i] \ \forall i$

Definition 3.4: (Tri-Component Partition Representation)

the system (4) is defined by:

A_N= (A^T, A^I , A^F) are the matrices containing the *T, I, F* components of A_N, respectively. $B_N = (B^T,\ B^I, B^F)$ are the vectors containing the corresponding components of B. $X_N = (X^T, X^I, X^F)$ are the Neutrosophic unknown vectors to be solved for each part. Where for all *i* = 1, 2..., m, *j* = 1, 2..., *n*:

A^T= $([T_{ij}^L, T_{ij}^U])$: Truth coefficient matrix,

A^I= $([I_{ij}^L, I_{ij}^U])$: Indeterminacy coefficient matrix,

A^F= $([F_{ij}^L, F_{ij}^U])$: falsity coefficient matrix,

$B^T = ([T_{b_i}^L, T_{b_i}^U])$: Truth right-hand vector,

$B^I = ([I_{b_i}^L, I_{b_i}^U])$: Indeterminacy right-hand vector,

$B^F = ([F_{b_i}^L, F_{b_i}^U])$: Falsity right-hand vector.

Therefor The system (4) can be decomposed into three associated interval-valued linear systems:

$$\begin{cases} A^T.X^T = B^T & truth\ (T) - \ system \\ A^I.X^I = B^I & indeterminacy\ (I) - \ system \\ A^F.X^F = B^F & falsity\ (F) - \ system \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (7)$$

Definition 3.5: (Tri-Component Determinants)

The determinants for each component are defined as follows:

Truth determinant: $\det([T_{ij}^L, T_{ij}^U])$

Indeterminacy determinant: $\det([I_{ij}^L, I_{ij}^U])$

Falsity determinant: $\det([F_{ij}^L, F_{ij}^U])$

4. Generalized Cramer's Rule for Tri-Component Interval-Valued Neutrosophic Linear Systems

In this section, we present a generalization of Cramer's rule specifically developed for tri- component interval-valued neutrosophic linear systems. The proposed generalization extends the classical rule to handle systems where the coefficients and constants are expressed as tri-component intervals (*T, I, F*). Following the derivation of the generalized formula, a solution analysis is conducted to examine the existence, uniqueness, or non-existence of solutions under various neutrosophic conditions. Finally, case studies for binary and ternary systems are provided to demonstrate the applicability and effectiveness of the proposed generalization through numerical examples.

Definition 4.1:

Consider a Neutrosophic Linear System consisting of n equations and n unknowns, where all coefficients and constant terms are expressed as tri-component interval-values. the system can be written as:

$$\begin{cases} ([T_{11}^L,\ T_{11}^U], [I_{11}^L, I_{11}^U], [F_{11}^L, F_{11}^U])x_1 \oplus ([T_{12}^L,\ T_{12}^U], [I_{12}^L, I_{12}^U], [F_{12}^L, F_{12}^U])x_2 \oplus \ldots \oplus \\ \quad ([T_{1n}^L,\ T_{1n}^U], [I_{1n}^L, I_{1n}^U], [F_{1n}^L, F_{1n}^U])x_n = ([T_{b_1}^L,\ T_{b_1}^U], [I_{b_1}^L, I_{b_1}^U], [F_{b_1}^L, F_{b_1}^U]) \\ ([T_{21}^L,\ T_{21}^U], [I_{21}^L, I_{21}^U], [F_{21}^L, F_{21}^U])x_1 \oplus ([T_{22}^L,\ T_{22}^U], [I_{22}^L, I_{22}^U], [F_{22}^L, F_{22}^U])x_2 \oplus \ldots \oplus \\ \quad ([T_{2n}^L,\ T_{2n}^U], [I_{2n}^L, I_{2n}^U], [F_{2n}^L, F_{2n}^U])x_n = ([T_{b_2}^L,\ T_{b_2}^U], [I_{b_2}^L, I_{b_2}^U], [F_{b_2}^L, F_{b_2}^U]) \\ \quad \vdots \qquad\qquad \vdots \qquad \ddots \qquad \vdots \\ ([T_{n1}^L,\ T_{n1}^U], [I_{n1}^L, I_{n1}^U], [F_{n1}^L, F_{n1}^U])x_1 \oplus ([T_{n2}^L,\ T_{n2}^U], [I_{n2}^L, I_{n2}^U], [F_{n2}^L, F_{n2}^U])x_2 \oplus \ldots \oplus \\ \quad ([T_{nn}^L,\ T_{nn}^U], [I_{nn}^L, I_{nn}^U], [F_{nn}^L, F_{nn}^U])x_n = ([T_{b_n}^L,\ T_{b_n}^U], [I_{n}^L, I_{b_n}^U], [F_{b_n}^L, F_{b_n}^U]) \end{cases} \quad \ldots\ldots\ldots\ldots \ (8)$$

Where ⊕ and . denoted the neutrosophic multiplication and Addition, respectively.

can be represented in the matrix form:
$A_N\, X_N = \; B_N$

where: $A_N = \left(\tilde{a}_{ij}\right)_{n\times n}$ is the coefficient matrix consisting of interval-valued neutrosophic elements, each entry being defined as:

$\tilde{a}_{ij} = ([T_{ij}^L,\; T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U\,])$. Here, $[T_{ij}^L, T_{ij}^U]$ represents the truth-membership interval, $[I_{ij}^L, I_{ij}^U]$ the indeterminacy-membership interval, and $[F_{ij}^L, F_{ij}^U\,]$ the falsity-membership interval. The vector of unknowns is given by $X_N = (x_1, x_2, \ldots\ldots, x_n)^{\mathrm{t}}$, while the output vector is expressed as $B_N = \left(\tilde{b}_i\right)_{n\times 1}$, with the each element defined as: $\tilde{b}_i = \; ([T_{b_i}^L, T_{b_i}^U], [I_{b_i}^L, I_{b_i}^U], [F_{b_i}^L, F_{b_i}^U]) \;\; \forall\, i = 1\; 2, \ldots, n.$

In This research relied on Cramer's rule as a mathematical analytical tool to solve the neutrosophic system. This method focused on finding the abstract mathematical solution to the system, where the coefficients *T, I,* and *F* were treated as pure mathematical values without imposing interval [0, 1] constraints during the solution process. It is worth noting that the obtained results represent the algebraic solution to the system, which may require a normalization process in subsequent applications to ensure that the values conform to the standard neutrosophic framework.

4.2: Cramer's Rule for Solving Tri-Component Interval-Valued Neutrosophic Linear Systems.

In this section, we present the algorithm for applying the generalized Cramer's Rule for solving tri-component interval-valued neutrosophic linear systems, including the computation of determinants in *T, I, F* components. The is applied following the steps:

4.2.1: Decompose the neutrosophic system into three independent crisp interval systems:

Truth (T) system: $\sum_{j=1}^{n}[T_{ij}^L,\; T_{ij}^U]\,.\; x_j \;= [T_{b_i}^L, T_{b_i}^U]\; \forall i$

Indeterminacy system (I): $\sum_{j=1}^{n}[I_{ij}^L, I_{ij}^U]\,.\; x_j \;= [I_{b_i}^L, I_{b_i}^U]\; \forall i$

Falsity system (F): $\sum_{j=1}^{n}[F_{ij}^L, F_{ij}^U]\,.\; x_j \;= [F_{b_i}^L, F_{b_i}^U]\; \forall i$

4.2.2: Calculate determinants for each system:

main neutrosophic determinant $\det_N(A_N)$ of the matrix A_N in (4)

is computed for each component *T, I, F.* as:

$\det_N(A_N) = (T_{\det_N(A_N)}, I_{\det_N(A_N)}, F_{\det_N(A_N)})$

$= \left(\left[T_{\det_N(A_N)}^L, T_{\det_N(A_N)}^U\right], [I_{\det_N(A_N)}^L, I_{\det_N(A_N)}^U]\,, [F_{\det_N(A_N)}^L, F_{\det_N(A_N)}^U]\right)$

Where:

$T_{\det_N(A_N)}^L = \det(T_{ij}^L)$ and $T_{\det_N(A_N)}^U = \det(\;T_{ij}^U),$

$I_{\det_N(A_N)}^L = \det(I_{ij}^L)$ and $I_{\det_N(A_N)}^U = \det(\;I_{ij}^U),$

$F_{\det_N(A_N)}^L = \det(F_{ij}^L)$ and $F_{\det_N(A_N)}^U = \det(\;F_{ij}^U).$

partial determinants $(\det(A_N)_{x_j})$.

For each unknown x_j, construct matrix ${A_N}^{(j)}$ by replacing the j-th column of A_N with the constant vector B_N, that is: ${A_N}^{(j)} = [Na_{j-1}, B_N, Na_{j+1} \ldots Na_n]$. Then

corresponding partial determinant is:

$det(A_N)_{x_j} = det\;({A_N}^{(j)}) = \left(\left[T_{det(A_N)_{x_j}}^L, T_{det(A_N)_{x_j}}^U\right]\;, \left[I_{det(A_N)_{x_j}}^L, I_{det(A_N)_{x_j}}^U\right], \left[F_{det(A_N)_{x_j}}^L, F_{det(A_N)_{x_j}}^U\right]\right) \forall j$

computed in the same way as the main determinant.

4.2.3: Computation of Solution Intervals

Calculate each unknown x_j using: $x_j = \det(A_N)_{x_j}/\det_N(A_N)$

Hence each component is obtained by:

$$x_j = \left(\frac{[T_{j\,det(A_N)_{x_j}}^{L},\ T_{j\,det(A_N)_{x_j}}^{U}]}{[T^{L}_{det_N(A_N)}, T^{U}_{det_N(A_N)}]},\ \frac{[I_{j\,det(A_N)_{x_j}}^{L},\ I_{j\,det(A_N)_{x_j}}^{U}]}{[I^{L}_{det_N(A_N)}, I^{U}_{det_N(A_N)}]},\ \frac{[F_{j\,det(A_N)_{x_j}}^{L}, F_{j\,det(A_N)_{x_j}}^{U}]}{[F^{L}_{det_N(A_N)}, F^{U}_{det_N(A_N)}]} \right)$$

$\forall j = 1,2, \ldots, n$

Division neutrosophic is well-defined under the non-zero lower-bound condition:

$T^{L}_{\det_N(A_N)} \neq 0, I^{L}_{\det_N(A_N)} \neq 0, F^{U}_{\det_N(A_N)} \neq 0$ (to avoid division by zero).

4.2.4: In the order to apply Cramer's rule to a tri-component interval-valued neutrosophic linear systems in (4), several key conditions must be satisfied to ensure solvability.

The system must first be square, meaning that the number of equations equals the number of variables. Furthermore, the Neutrosophic determinant ($\det_N(A_N) \neq \tilde{0}$) should be non-zero, which guarantees the structural solvability of the system. Specifically, the truth component should have a positive upper bound, the indeterminacy component should not exceed an upper limit of one, the falsity component should have a Lower bound less than one. Additionally, the stability of Neutrosophic values is essential; this requires that the maximum upper bound of falsity does not surpass the minimum lower bound of truth, and the maximum upper bound of indeterminacy does not exceed one minus the minimum lower bound of truth. Meeting these criteria that the system is mathematically suitable for the application of Cramer's rule within the Neutrosophic framework.

4.2.5: Solvability Conditions.

This part investigates the conditions under which tri-component interval-valued neutrosophic linear systems admit solutions. These conditions provide a formal criterion to classify the solution space and crucial for interpreting the neutrosophic systems behavior under uncertainty, as summarized in table 5.

Remark 4.2: It is worth noting that neutrosophic solutions may include negative values or values exceeding 1. This phenomenon does not invalidate the solution under the generalized Cramer's rule, provided that the neutrosophic determinant remains non-zero. Instead, it highlights the extended generality of Neutrosophic theory, which operates beyond the classical [0, 1] interval and allows for a broader representation of truth, indeterminacy, and falsity.

Table 5. Solvability conditions for tri-component interval-valued neutrosophic linear systems.

Solution Type	Mathematical Conditions	Interpretation in (T, I, F)
Unique solution	$\det_N(A_N) \neq \tilde{0}$	All T, I, F components are consistent and intervals are well-defined, leading to a single unique solution.
Infinite Solutions	$\det_N(A_N) = \tilde{0}, \det(A_j) \neq 0\ \forall\ j$	system is undetermined; solution intervals overlap, resulting in infinite many solutions.
No solution	$\det_N(A_N) = \tilde{0}$, $\det(A_j) = 0$ for some j	Intervals are inconsistent across components, leading to a contradictory and unsolvable system.

4.3 Binary Tri-Component Interval-Valued Neutrosophic Linear Systems

This section, applies the algorithm to 2×2 systems, with detailed solution analysis for cases of unique solution, infinite solutions, and no solution.

Definition 4.3: Consider binary tri-component interval-valued neutrosophic linear systems (2×2) it is written as follows:

$$\begin{cases} ([T_{11}^L,\ T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U])x \oplus ([T_{12}^L,\ T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U])y \\ \qquad = ([T_{b_1}^L,\ T_{b_1}^U],[I_{b_1}^L,I_{b_1}^U],[F_{b_1}^L,F_{b_1}^U]) \\ ([T_{21}^L,\ T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U])x \oplus ([T_{22}^L,\ T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U])y \\ \qquad = ([T_{b2}^L,\ T_{b2}^U],[I_{b2}^L,I_{b2}^U],[F_{b2}^L,F_{b2}^U]) \end{cases} \quad \ldots\ldots\ldots (9)$$

It can be represented in matrix form as:

$$A_N\, X_N = B_N$$

where:

$$A_N = \begin{pmatrix} ([T_{11}^L,\ T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U]) & ([T_{12}^L,\ T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U]) \\ ([T_{21}^L,\ T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U]) & ([T_{22}^L,\ T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U]) \end{pmatrix},\ X_N = \begin{pmatrix} x \\ y \end{pmatrix},\ B_N = \begin{pmatrix} ([T_{b1}^L,\ T_{b1}^U],[I_{b1}^L,I_{b1}^U],[F_{b1}^L,F_{b1}^U]) \\ ([T_{b2}^L,\ T_{b2}^U],[I_{b2}^L,I_{b2}^U],[F_{b2}^L,F_{b2}^U]) \end{pmatrix} \ldots\ldots\ldots (10)$$

4.3.1: Solve the system (9)

4.3.1.1: Compute the main neutrosophic determinant of the matrix A_N in (10) is given by:

$$\det_N(A_N) = \left(\det_N(T), \det_N(I), \det_N(F)\right) =$$

$$\begin{vmatrix} ([T_{11}^L,T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U]) & ([T_{12}^L,T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U]) \\ ([T_{21}^L,T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U]) & ([T_{22}^L,\ T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U]) \end{vmatrix}$$

$$= \left(\begin{vmatrix} [T_{11}^L,T_{11}^U] & [T_{12}^L,T_{12}^U] \\ [T_{21}^L,T_{21}^U] & [T_{22}^L,T_{22}^U] \end{vmatrix}, \begin{vmatrix} [I_{11}^L,I_{11}^U] & [I_{12}^L,I_{12}^U] \\ [I_{21}^L,I_{21}^U] & [I_{22}^L,I_{22}^U] \end{vmatrix}, \begin{vmatrix} [F_{11}^L,F_{11}^U] & [F_{12}^L,F_{12}^U] \\ [F_{21}^L,F_{21}^U] & [F_{22}^L,F_{22}^U] \end{vmatrix} \right)$$

$$= \left(([T_{11}^L,T_{11}^U][T_{22}^L,T_{22}^U] - [T_{12}^L,T_{12}^U][T_{21}^L,T_{21}^U]),\ ([I_{11}^L,I_{11}^U][I_{22}^L,I_{22}^U] - [I_{12}^L,I_{12}^U]\,[I_{21}^L,I_{21}^U]),\ ([F_{11}^L,F_{11}^U][F_{22}^L,F_{22}^U] - [F_{12}^L,F_{12}^U][F_{21}^L,F_{21}^U]) \right).$$

4.3.1.2: Partial determinants for the unknowns are computed as follows:

For x : $\det(A_N)_x = (\det(T_x), \det(I_x), \det(F_x))$

$$= \left(\begin{vmatrix} [T_{b_1}^L,T_{b_1}^U] & [T_{12}^L,T_{12}^U] \\ [T_{b_2}^L,T_{b_2}^U] & [T_{22}^L,T_{22}^U] \end{vmatrix}, \begin{vmatrix} [I_{b_1}^L,I_{b_1}^U] & [I_{12}^L,I_{12}^U] \\ [I_{b_2}^L,I_{b_2}^U] & [I_{22}^L,I_{22}^U] \end{vmatrix}, \begin{vmatrix} [F_{b_1}^L,F_{b_1}^U] & [F_{12}^L,F_{12}^U] \\ [F_{b_2}^L,F_{b_2}^U] & [F_{22}^L,F_{22}^U] \end{vmatrix} \right)$$

$$= (([T_{b_1}^L,T_{b_1}^U][T_{22}^L,T_{22}^U] - [T_{12}^L,T_{12}^U][T_{b_2}^L,T_{b_2}^U]),\ ([I_{b_1}^L,I_{b_1}^U]\,[I_{22}^L,I_{22}^U] - [I_{12}^L,I_{12}^U]\,[I_{b_2}^L,I_{b_2}^U]), ([F_{b_1}^L,F_{b_1}^U][F_{22}^L,F_{22}^U] - [F_{12}^L,F_{12}^U][F_{b_2}^L,F_{b_2}^U])).$$

For y : $\det(A_N)_y = \left(\det(T_y), \det(I_y), \det(F_y)\right)$

$$= \left(\begin{vmatrix} [T_{11}^L,T_{11}^U] & [T_{b_1}^L,T_{b_1}^U] \\ [T_{21}^L,T_{21}^U] & [T_{b_2}^L,T_{b_2}^U] \end{vmatrix}, \begin{vmatrix} [I_{11}^L,I_{11}^U] & [I_{b_1}^L,I_{b_1}^U] \\ [I_{21}^L,I_{21}^U] & [I_{b_2}^L,I_{b_2}^U] \end{vmatrix}, \begin{vmatrix} [F_{11}^L,F_{11}^U] & [F_{b_1}^L,F_{b_1}^U] \\ [F_{21}^L,F_{21}^U] & [F_{b_1}^L,F_{b_1}^U] \end{vmatrix} \right)$$

$$= (([T_{11}^L,T_{11}^U][T_{b_2}^L,T_{b_2}^U] - [T_{b_1}^L,T_{b_1}^U]\,[T_{21}^L,T_{21}^U]), (\,[I_{11}^L,I_{11}^U]\,[I_{b_2}^L,I_{b_2}^U] - [I_{b_1}^L,I_{b_1}^U]\,[I_{21}^L,I_{21}^U]), ([F_{11}^L,F_{11}^U][F_{b_2}^L,F_{b_2}^U] - [F_{b_1}^L,F_{b_1}^U][F_{21}^L,F_{21}^U]))$$

4.3.1.3: Solving of the system (9):

The unknowns are determined using the Neutrosophic division as follows:

$x = \det(A_N)_x / \det_N(A_N) =$

$$\Big(\frac{([T^L_{b_1}, T^U_{b_1}][T^L_{22}, T^U_{22}] - [T^L_{12}, T^U_{12}][T^L_{b_2}, T^U_{b_2}])}{([T^L_{11}, T^U_{11}][T^L_{22}, T^U_{22}] - [T^L_{12}, T^U_{12}][T^L_{21}, T^U_{21}])}, \frac{([I^L_{b_1}, I^U_{b_1}]\,[I^L_{22}, I^U_{22}] - [I^L_{12}, I^U_{12}]\,[I^L_{b_2}, I^U_{b_2}])}{([I^L_{11}, I^U_{11}][I^L_{22}, I^U_{22}] - [I^L_{12}, I^U_{12}]\,[I^L_{21}, I^U_{21}])},$$

$$\frac{([F^L_{b_1}, F^U_{b_1}][F^L_{22}, F^U_{22}] - [F^L_{12}, F^U_{12}\,][F^L_{b_2}, F^U_{b_2}\,])}{([F^L_{11}, F^U_{11}\,][F^L_{22}, F^U_{22}] - [F^L_{12}, F^U_{12}\,][F^L_{21}, F^U_{21}\,])}\Big) = (x_T\,, x_I, x_F),$$

$y = \det(A_N)_y / \det_N(A_N) =$

$$\Big(\frac{(\,[T^L_{11}, T^U_{11}][T^L_{b_2}, T^U_{b_2}] - [T^L_{b_1}, T^U_{b_1}]\,[T^L_{21}, T^U_{21}])}{([T^L_{11}, T^U_{11}][T^L_{22}, T^U_{22}] - [T^L_{12}, T^U_{12}][T^L_{21}, T^U_{21}])}, \frac{(\,[I^L_{11}, I^U_{11}]\,[I^L_{b_2}, I^U_{b_2}] - [I^L_{b_1}, I^U_{b_1}]\,[I^L_{21}, I^U_{21}])}{([I^L_{11}, I^U_{11}][I^L_{22}, I^U_{22}] - [I^L_{12}, I^U_{12}]\,[I^L_{21}, I^U_{21}])},$$

$$\frac{([F^L_{11}, F^U_{11}\,][F^L_{b_2}, F^U_{b_2}] - [F^L_{b_1}, F^U_{b_1}][F^L_{21}, F^U_{21}\,])}{([F^L_{11}, F^U_{11}\,][F^L_{22}, F^U_{22}] - [F^L_{12}, F^U_{12}\,][F^L_{21}, F^U_{21}\,])}\Big) = (y_T\,, y_I, y_F)$$

*4.3.1.4: The final solution of **a binary** neutrosophic system (9) via Cramer's rule:*

Is expressed in terms of the truth component, indeterminacy component and falsity component as follows:

$$x = \left(\frac{\left[T^L_{\det(A_N)_x}\ T^U_{\det(A_N)_x}\right]}{\left[T^L_{\det_N(A_N)} T^U_{\det_N(A_N)}\right]}, \frac{\left[I^L_{\det(A_N)_x}\ I^U_{\det(A_N)_x}\right]}{\left[I^L_{\det_N(A_N)}\ I^U_{\det_N(A_N)}\right]}, \frac{\left[F^L_{\det(A_N)_x}\ F^U_{\det(A_N)_x}\right]}{\left[F^L_{\det_N(A_N)}\ F^U_{\det_N(A_N)}\right]}\right),$$

$$y = \left(\frac{[T^L_{\det(A_N)_y}\ T^U_{\det(A_N)_y}]}{[T^L_{\det_N(A_N)}\ T^U_{\det_N(A_N)}]}, \frac{[I^L_{\det(A_N)_y}\ I^U_{\det(A_N)_y}]}{[I^L_{\det_N(A_N)}\ I^U_{\det_N(A_N)}]}, \frac{[F^L_{\det(A_N)_y}\ F^U_{\det(A_N)_y}]}{[F^L_{\det_N(A_N)}\ F^U_{\det_N(A_N)}]}\right).$$

4.4: Numerical Examples

Example 1: Let

$$\begin{cases} ([0.1,0.2], [0.1, 0.2], [0.3,0.4\,])x \oplus ([0.3, 0.4], [0.2, 0.3], [0.1, 0.2\,])y \\ \qquad = ([0.7, 0.9], [0.3, 0.5], [0.5, 0.7\,]) \\ ([0.2, 0.3], [0.0, 0.1], [0.2, 0.3\,])x \oplus ([0.1, 0.2], [0.1, 0.2], [0.4, 0.5\,])y \\ \qquad = ([0.8, 1.0], [0.1,0.3], [0.6, 0.8\,]) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Find solution of the system.

Solution: It can be represented in matrix form as: $A_N\, X_N = B_N$, where:

$$\begin{pmatrix} ([0.1, 0.2], [0.1, 0.2], [0.3,0.4\,]) & ([0.3, 0.4], [0.2, 0.3], [0.1, 0.2\,]) \\ ([0.2, 0.3], [0.0, 0.1], [0.2, 0.3\,]) & ([0.1,0.2], [0.1, 0.2], [0.4, 0.5\,]) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} ([0.7, 0.9], [0.3, 0.5], [0.5, 0.7\,]) \\ ([0.8, 1], [0.1,0.3], [0.6, 0.8\,]) \end{pmatrix}$$

4.4.1: Compute the main neutrosophic determinant:

$$det_N(A_N) = \begin{vmatrix} ([0.1, 0.2], [0.1, 0.2], [0.3,0.4\,]) & ([0.3, 0.4], [0.2, 0.3], [0.1, 0.2\,]) \\ ([0.2, 0.3], [0.0, 0.1], [0.2, 0.3\,]) & ([0.1, 0.2], [0.1, 0.2], [0.4, 0.5\,]) \end{vmatrix}$$

4.4.2: Decompose the system into matrices (T, I, F)

Truth matrix(A^T): $\begin{pmatrix} [0.1, 0.2] & [0.3, 0.4] \\ [0.2, 0.3] & [0.1, 0.2] \end{pmatrix}$

Indeterminacy matrix(A^I): $\begin{pmatrix} [0.1, 0.2] & [0.2, 0.3] \\ [0.0, 0.1] & [0.1, 0.2] \end{pmatrix}$

Falsity matrix(A^F): $\begin{pmatrix} [0.3,0.4\ ] & [0.1, 0.2\ ] \\ [0.2, 0.3\ ] & [0.4, 0.5\ ] \end{pmatrix}$

4.4.3: Calculate the main determinants of the system (11)

a) Truth component determinant (T): $\det_N(T)$ =

$$\begin{vmatrix} [0.1, 0.2] & [0.3, 0.4] \\ [0.2, 0.3] & [0.1, 0.2] \end{vmatrix}$$

= (([0.1, 0.2][0.1, 0.2] − [0.3, 0.4][0.2, 0.3]) = ([0.01,.0.04]-[0.06,0.12] = [-0.023, -0.021]

b) Indeterminacy component(I): $\det_N(I)$ = $\begin{vmatrix} [0.1, 0.2] & [0.2, 0.3] \\ [0.0, 0.1] & [0.1, 0.2] \end{vmatrix}$

= [0.1, 0.2][0.1, 0.2] − [0.2, 0.3][0.0, 0.1]

= [0.19, 0. 36] - [0.2,0.37] = [0.514,1.8],

c) Falsity component(F): $\det_N(F)$ = $\begin{vmatrix} [0.3,0.4\ ] & [0.1, 0.2\ ] \\ [0.2, 0.3\ ] & [0.4, 0.5\ ] \end{vmatrix}$

= [0.3,0.4][0.4, 0.5] − [0.1, 0.2][0.2, 0.3]

= *[0.58, 0.7] – [0.28,0.44] = [1.32,2.5]. Hence*

$det_N(A_N) = (det_N(T), det_N(I), det_N(F))$

$= \left(\begin{vmatrix} [1,2] & [3,4] \\ [2,3] & [1,2] \end{vmatrix}, \begin{vmatrix} [0.1, 0.2] & [0.2, 0.3] \\ [0.0, 0.1] & [0.1, 0.2] \end{vmatrix}, \begin{vmatrix} [0.3,0.4\ ] & [0.1, 0.2\ ] \\ [0.2, 0.3\ ] & [0.4, 0.5\ ] \end{vmatrix} \right)$

= *([-0.023, -0.021], [0.514,1.8], [1.32,2.5])*

4.4.4: Calculation of Partial determinants for the variables unknowns ($det(A_N)_x$, $det(A_N)_y$):

For x : $\det(A_N)_x = (\det(T_x), \det(I_x), \det(F_x))$

$= \left(\begin{vmatrix} [0.7, 0.9] & [0.3,0.4] \\ [0.8, 1.0] & [0.1, 0.2] \end{vmatrix}, \begin{vmatrix} [0.3, 0.5] & [0.2, 0.3] \\ [0.1,0.3] & [0.1, 0.2] \end{vmatrix}, \begin{vmatrix} [0.5, 0.7\ ] & [0.1, 0.2\ ] \\ [0.6, 0.8\ ] & [0.4, 0.5\ ] \end{vmatrix} \right)$

= ([0.7, 0.9][0.1, 0.2] − [0.3, 0.4][0.8, 1.0], [0.3, 0.5][0.1, 0.2] − [0.2, 0.3][0.1,0.3],

[0.5, 0.7][0.4, 0.5] − [0.1, 0.2][0.6, 0.8])

= ([0.7,0.18] − [0.24,0.4], [0.37,0.6] − [0.28,0.51], [0.7,0.85] − [0.64,0.84])

= ([0.5, −0.08], [0.73,2.14], [0.83,1.33])

For y : $det(A_N)_y = \left(det(T_y), det(I_y), det(F_y)\right)$

$= \left(\begin{vmatrix} [0.1, 0.2] & [0.7, 0.9] \\ [0.2, 0.3] & [0.8, 1.0] \end{vmatrix}, \begin{vmatrix} [0.1, 0.2] & [0.3, 0.5] \\ [0.0, 0.1] & [0.1, 0.3] \end{vmatrix}, \begin{vmatrix} [0.3, 0.4] & [0.5, 0.7] \\ [0.2, 0.3] & [0.6, 0.8] \end{vmatrix} \right)$

$= ([0.1, 0.2][0.8, 1.0]$ -$[0.7, 0.9][0.2, 0.3], [0.1, 0.2][0.1, 0.3] - [0.3, 0.5][0.0, 0.1],$

$[0.3, 0.4][0.6, 0.8] - [0.5, 0.7][0.2, 0.3])$

= ([0.08,0.2]- [0.14,0.27], [0.19,0.44]- [0.3,0.55], [0.72,0.88]- [0.6,0.79])

= ([-0.26,0.07], [0.35, 1.47], [0.91,1.47]).

4.4.5: The solution of a binaray neutrosophic system via Cramer's rule is expressed in terms of the truth, indeterminacy and falsity as follows:

$x = det(A_N)_x / det_N(A_N), \quad y = det(A_N)_y / det_N(A_N)$

$= \left(\frac{[0.5, -0.08]}{[-0.023, -0.021]}, \frac{[0.73, 2.14]}{[0.514, 1.8]}, \frac{[0.83, 1.33]}{[1.32, 2.5]} \right)$

$= ([-23.81, 3.48], [1.34, 3.35], [1.11, -0.03]),$

$y = \left(\frac{[-0.26, 0.07]}{[-0.023, -0.021]}, \frac{[0.35, 1.47]}{[0.514, 1.8]}, \frac{[0.91, 1.47]}{[1.32, 2.5])} \right)$

$= ([12.381, -3.04], [1.81, 1.97], [1.07, -0.47]).$

Therefore, the final solution to the nitrosophy system (according to Cramer's rule mathematically) is given by:

$S = \{(x, y) \in R^2 \cup \{I\}: x = ([-23.81, 3.48], [1.34, 3.35], [1.11, -0.03]),$

$y = ([12.381, -3.04], [1.81, 1.97], [1.07, -0.47])\}.$

The results derived using Cramer's rule demonstrate the exact mathematical solution for the studied system. Since this method maintains the mathematical integrity of the processes without restriction. Some values outside the range [0,1] appear. This does not reduce the accuracy of the solution, but rather reflects the nature of the analytical approach used. These solutions can be considered the basis for subsequent studies that include advanced normalization mechanisms to convert them into practically applicable neutrosophic formulas.

Mathematical Verification the unique solution by substitution: Taking values of the solution $x = (x_T, x_I, x_F), y = (y_T, y_I, y_F)$ and from definition 2.3 we get:

Equation 1:

$([0.1, 0.2], [0.1, 0.2], [0.3, 0.4])x \oplus ([0.3, 0.4], [0.2, 0.3], [0.1, 0.2])y =$

$([0.1, 0.2], [0.1, 0.2], [0.3, 0.4])([-23.81, 3.48], [1.34, 3.35], [1.11, -0.03]) \oplus$

$([0.3, 0.4], [0.2, 0.3], [0.1, 0.2])([12.381, -3.04], [1.81, 1.97], [1.07, -0.47])$

$= ([-2.38, 0.70], [1.306, 2.88], [1.08, 0.382]) \oplus$ *([3.71, -1.216], [1.65,1.68], [1.06,0.576]*

$= ([10.16, 0.34], [2.15, 4.84], [1.15, 0.22]) \neq ([0.7, 0.9], [0.3, 0.5], [0.5, 0.7])$

The essential nature of neutrosophic systems have unique algebraic properties that are fundamentally different from classical systems. While classical mathematics aims to find exact and absolute solutions, neutrosophic logic recognizes the uncertain nature of real-world data.

There for the mismatch in the substitution is not a computational error. But rather an accurate mathematical reflection of the uncertainty present in the original data. If the data are ambiguous, it stands to reason that the solutions will also be ambiguous.

In practical applications (such as opinion polls, weather forecasts, and economic analysis), data are rarely completely accurate. Neutrosophic logic recognizes this fact and addresses it explicitly. The degree of mismatch can be viewed as measure of the degree of ambiguity in a system. The greater the mismatch, the greater the inherent uncertainty in the data and equations. ([1], [22]).

Example 2. Let

$$\begin{cases}([0.1,0.1],[0,0],[0,0])x \oplus ([0.1,0.1],[0,0],[0,0])y = ([0.2,0.2],[0,0],[0,0]) \\ ([0.1,0.1],[0,0],[0,0])x \oplus ([0.1,0.1],[0,0],[0,0])y = ([0.3,0.3],[0,0],[0,0]) \end{cases} \cdots\cdots\cdots\cdots\cdots\cdots (12)$$

Find solution of the system.

Solution: $A_N = \begin{pmatrix} ([0.1,0.1],[0,0],[0,0]) & ([0.1,0.1],[0,0],[0,0]) \\ ([0.1,0.1],[0,0],[0,0]) & ([0.1,0.1],[0,0],[0,0]) \end{pmatrix}$,

$$det_N(A_N) = \begin{vmatrix} ([0.1,0.1],[0,0],[0,0]) & ([0.1,0.1],[0,0],[0,0]) \\ ([0.1,0.1],[0,0],[0,0]) & ([0.1,0.1],[0,0],[0,0]) \end{vmatrix}$$

$$= \left(\begin{vmatrix} [0.1,0.1] & [0.1,0.1] \\ [0.1,0.1] & [0.1,0.1] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix} \right)$$
$$= ([0,0],[0,0],[0,0]),$$

$$\det(A_N)_x = (\det(T_x), \det(I_x), \det(F_x))$$

$$= \left(\begin{vmatrix} [0.2,0.2] & [0.1,0.1] \\ [0.3,0.3] & [0.1,0.1] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix} \right)$$

$$= ([0.2,0.2][0.1,0.1] - [0.1,0.1][0.3,0.3], [0,0], [0,0])$$

$$= ([0.02,0.02] - [0.03,0.03], [0,0], [0,0])$$

$$= ([-0.01, -0.01], [0,0], [0,0]),$$

$$det(A_N)_y = \left(det(T_y), det(I_y), det(F_y)\right)$$

$$= \left(\begin{vmatrix} [0.1,0.1] & [0.2,0.2] \\ [0.1,0.1] & [0.3,0.3] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix}, \begin{vmatrix} [0,0] & [0,0] \\ [0,0] & [0,0] \end{vmatrix} \right) = ([0.01, 0.01], [0,0], [0,0]),$$

Neutrosophic Result: Since the neutrosophic determinant is $\det_N(A_N) = \tilde{0}$ while $\det(A_N)_x \neq \tilde{0}$ and $\det(A_N)_y \neq \tilde{0}$, the considered neutrosophc linear system is inconsistent, and therefore has no solution.

Example 3. Let

$$\begin{cases}([0.1,0.1],[0,0],[0,0])x \oplus ([0.1,0.1],[0,0],[0,0])y = ([0.2,0.2],[0,0],[0,0]) \\ ([0.2,0.2],[0,0],[0,0])x \oplus ([0.2,0.2],[0,0],[0,0])y = ([0.4,0.4],[0,0],[0,0]) \end{cases} \cdots\cdots\cdots\cdots\cdots\cdots (13)$$

Find solution of the system.

Solution: $A_N = \begin{pmatrix} ([0.1,0.1],[0,0],[0,0]) & ([0.1,0.1],[0,0],[0,0]) \\ ([0.2,0.2],[0,0],[0,0]) & ([0.2,0.2],[0,0],[0,0]) \end{pmatrix}$

$det_N(A_N) = \begin{vmatrix} ([0.1, 0.1], [0, 0], [0,0]) & ([0.1, 0.1], [0, 0], [0,0]) \\ ([0.2, 0.2], [0, 0], [0,0]) & ([0.2, 0.2], [0, 0], [0,0]) \end{vmatrix}$

$= \left(\begin{vmatrix} [0.1,0.1] & [0.1, 0.1] \\ [0.2,0.2] & [0.2,0.2] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix} \right) = ([0, 0], [0, 0], [0, 0]),$

$det(A_N)_x = (\begin{vmatrix} [0.2,0.2] & [0.1, 0.1] \\ [0.4, 0.4] & [0.2,0.2] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix})$

$= ([0, 0], [0, 0], [0, 0]),$

$det(A_N)_y = \left(\begin{vmatrix} [0.1, 0.1] & [0.2,0.2] \\ [0.2,0.2] & [0.4, 0.4] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix}, \begin{vmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{vmatrix} \right)$

$= ([0, 0], [0, 0], [0, 0]),$

Since, $\det_N(A_N) = \det(A_N)_x = \det(A_N)_y = \tilde{0}$, then the system has an infinite number of neutrosophic solution. Therefor if we assume that:

$y = (t_T, t_I, t_F) = ([t, t], [t, t], [t, t])$

any free variable then y with all its components *T, I, F* is directly related to the parameter.

$([0.1, 0.1], [0, 0], [0,0])x \oplus ([0.1, 0.1], [0, 0], [0,0])t = ([0.2, 0.2], [0, 0], [0,0])$

$([0.1, 0.1], [0, 0], [0,0])x = ([0.2, 0.2], [0, 0], [0,0]) - ([0.1, 0.1], [0, 0], [0,0])$

$([t, t], [t, t], [t, t])$. *Since,* $x = (x_T , x_I, x_F)$ *then* $[0.1, 0.1]x_T = [0.2,0.2] - [0.1,0.1][t, t]$. *From definition: 2.3 we get:*
$x_T = \dfrac{[0.2,0.2] - [0.1,0.1][t, t]}{[0.1, 0.1]}$, $x_I = [0,0]$, $x_F = [0,0]$.

Henec The solution of a neutrosophic system is:

$$x = \left(\frac{[0.2 - t_T, 0.2 - t_T]}{[0.1, 0.1]}, [0,0], [0,0] \right), y = (t_T, t_I, t_F).$$

Thus, the system has an infinite number of neutrosophic solution. Therefor the solution is given by:

$$S = \{ \left(\frac{[0.2 - t_T, 0.2 - t_T]}{[0.1, 0.1]}, [0,0], [0,0] \right), (t_T, t_I, t_F);\ t \in R \cup \{I\}\}$$

Example 4. Let

$$\begin{cases} ([0.5, 0.6], [0.6,0.7], [0.8,0.9])x \oplus ([0.1, 0.2], [0.2, 0.3], [0.3,0.4])y = ([0, 0], [0, 0], [0,0]) \\ ([0.1, 0.2], [0.2, 0.3], [0.3,0.4])x \oplus ([0.5, 0.6], [0.6, 0.7], [0.8,0.9])y = ([0, 0], [0,0], [0, 0]) \end{cases} \dots\dots\dots\dots\dots (14)$$

Find solution of the system.

Solution: $A_N = \begin{pmatrix} ([0.5, 0.6], [0.6, 0.7], [0.8,0.9]) & ([0.1,0.2], [0.2,0.3], [0.3,0.4]) \\ ([0.1, 0.2], [0.2, 0.3], [0.3,0.4]) & ([0.5, 0.6], [0.6, 0.7], [0.8,0.9]) \end{pmatrix}$

$det_N(A_N) = \begin{vmatrix} ([0.5, 0.6], [0.6, 0.7], [0.8,0.9]) & ([0.1,0.2], [0.2,0.3], [0.3,0.4]) \\ ([0.1, 0.2], [0.2, 0.3], [0.3,0.4]) & ([0.5, 0.6], [0.6, 0.7], [0.8,0.9]) \end{vmatrix}$

$= \left(\begin{vmatrix} [0.5, 0.6] & [0.1, 0.2] \\ [0.1, 0.2] & [0.5, 0.6] \end{vmatrix}, \begin{vmatrix} [60., 0.7] & [0.2, 0.3] \\ [0.2, 0.3] & [0.6, 0.7] \end{vmatrix}, \begin{vmatrix} [0.8,0.9] & [0.3,0.4] \\ [0.3,0.4] & [0.8,0.9] \end{vmatrix} \right)$

$= (([0.5, 0.6][0.5, 0.6] - [0.1, 0.2][0.1, 0.2]),$

$$([0.6,0.7][0.6,0.7]-[0.2,0.3][0.2,0.3]),([0.8,0.9][0.8,0.9]-[0.3,0.4][0.3,0.4]))$$

$= ([0.25,0.36]-[0.01,0.04],[0.84,0.91]-[0.36,0.51],[0.96,0.99]-[0.51,0.64])$

= ([0.22, 0.37], [$1.65,2.53], [1.5,1.94]$*). Based on the calculation of the neutrosophic determinant of the matrix* A_N*, where:*

$\det_N(A_N) = ([0.22,0.37],[1.65,2.53],[1.5,1.94])$, we observe that values of the components (T, I, F) do not include zero (as all lower and upper bounds are strictly positive). This indicates that the determinant is not equal to the Neutrosophic zero $([0,0],[0,0],[0,0])$([20], [22]). Consequently, the homogeneous system $A_N X_N = \tilde{0}$ has only a trivial solution i.e., $X_N = ([0,0],[0,0],[0,0])$.

4.4: Ternary Tri-Component Interval-Valued Neutrosophic Linear Systems (3×3):

This section demonstrates the use of the generalized rule for 3×3 systems, presenting analytical solutions and discussing existence and uniqueness conditions with illustrative numerical examples.

Definition 4.4: The Neutrosophic ternary system can be represented as a set of three equations in three unknowns, in which each coefficient and constant term is expressed as a tri-component closed interval. Corresponding to the degree of truth, inderminacy, and falsity. the system can be written as:

$$\begin{cases} ([T_{11}^L, T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U])x \oplus ([T_{12}^L, T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U])y \\ \oplus([T_{13}^L, T_{13}^U],[I_{13}^L,I_{13}^U],[F_{13}^L,F_{13}^U])z = ([T_{b1}^L, T_{b1}^U],[I_{b1}^L,I_{b1}^U],[F_{b1}^L,F_{b1}^U]) \\ ([T_{21}^L, T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U])x \oplus ([T_{22}^L, T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U])y \\ \oplus([T_{23}^L, T_{23}^U],[I_{23}^L,I_{23}^U],[F_{23}^L,F_{23}^U])z = ([T_{b2}^L, T_{b2}^U],[I_{b2}^L,I_{b2}^U],[F_{b2}^L,F_{b2}^U]) \\ ([T_{31}^L, T_{31}^U],[I_{31}^L,I_{31}^U],[F_{31}^L,F_{31}^U])x \oplus ([T_{32}^L, T_{32}^U],[I_{32}^L,I_{32}^U],[F_{32}^L,F_{32}^U])y \\ \oplus([T_{33}^L, T_{33}^U],[I_{33}^L,I_{33}^U],[F_{33}^L,F_{33}^U])z = ([T_{b3}^L, T_{b3}^U],[I_{b3}^L,I_{b3}^U],[F_{b3}^L,F_{b3}^U]) \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (15)$$

Where the operations : $\oplus$ and $.$ denote neutrosophic addition and multiplication, respectively. It can be represented in matrix form as:

$A_N \, X_N = B_N$

where:

$$A_N = \begin{pmatrix} ([T_{11}^L, T_{11}^U],[I_{11}^L,I_{11}^U],[F_{11}^L,F_{11}^U]) & ([T_{12}^L, T_{12}^U],[I_{12}^L,I_{12}^U],[F_{12}^L,F_{12}^U]) & ([T_{13}^L, T_{13}^U],[I_{13}^L,I_{13}^U],[F_{13}^L,F_{13}^U]) \\ ([T_{21}^L, T_{21}^U],[I_{21}^L,I_{21}^U],[F_{21}^L,F_{21}^U]) & ([T_{22}^L, T_{22}^U],[I_{22}^L,I_{22}^U],[F_{22}^L,F_{22}^U]) & ([T_{23}^L, T_{23}^U],[I_{23}^L,I_{23}^U],[F_{23}^L,F_{23}^U]) \\ ([T_{31}^L, T_{31}^U],[I_{31}^L,I_{31}^U],[F_{31}^L,F_{31}^U]) & ([T_{32}^L, T_{32}^U],[I_{32}^L,I_{32}^U],[F_{32}^L,F_{32}^U]) & ([T_{33}^L, T_{33}^U],[I_{33}^L,I_{33}^U],[F_{33}^L,F_{33}^U]) \end{pmatrix}$$

$$X_N = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B_N = \begin{pmatrix} ([T_{b_1}^L, T_{b_1}^U],[I_{b_1}^L,I_{b_1}^U],[F_{b_1}^L,F_{b_1}^U]) \\ ([T_{b_2}^L, T_{b_2}^U],[I_{b_2}^L,I_{b_2}^U],[F_{b_2}^L,F_{b_2}^U]) \\ ([T_{b_3}^L, T_{b_3}^U],[I_{b_3}^L,I_{b_3}^U],[F_{b_3}^L,F_{b_3}^U]) \end{pmatrix}$$

4.4.1: Solve the system (15):

The main determinant $\det_N(A_N)$ for the 3×3 coefficient matrix A_N is defined as:

$$det_N(A_N) = (T_{det_N(A_N)}, I_{det_N(A_N)}, F_{det_N(A_N)})$$

which is obtained by calcining separately the determinants of the truth, inderminacy, and falsity sub-matrices. Similarly, auxiliary determinants for each variable are computed by replacing the corresponding column with the Neutrosophic constants, yielding:

$For\ x : det_N(A_N)_x = (det(T_x), det(I_x), det(F_x))$,

$For\ y : det_N(A_N)_y = \left(det(T_y), det(I_y), det(F_y)\right),$

$For\ z : det_N(A_N)_z = (det(T_z), det(I_z), det(F_z)).$

Final Solution for system (15):

The unknowns are determined using the Neutrosophic division as follows:

$x = \det(A_N)_x/\det_N(A_N) = (x_T, x_I, x_F)$:

$$For\ truth\ component\ of\ x:\ x_T = \frac{det\begin{pmatrix}[T^L_{b1},T^U_{b1}] & [T^L_{12},T^U_{12}] & [T^L_{13},T^U_{13}]\\ [T^L_{b_2},T^U_{b_2}] & [T^L_{22},T^U_{22}] & [T^L_{23},T^U_{23}]\\ [T^L_{b_3},T^U_{b_3}] & [T^L_{32},T^U_{32}] & [T^L_{33},T^U_{33}]\end{pmatrix}}{det\begin{pmatrix}[T^L_{11},T^U_{11}] & [T^L_{12},T^U_{12}] & [T^L_{13},T^U_{13}]\\ [T^L_{21},T^U_{21}] & [T^L_{22},T^U_{22}] & [T^L_{23},T^U_{23}]\\ [T^L_{31},T^U_{31}] & [T^L_{32},T^U_{32}] & [T^L_{33},T^U_{33}]\end{pmatrix}},$$

$$For\ inderminacy\ component\ of\ x:\ x_I = \frac{det\begin{pmatrix}[T^L_{11},T^U_{11}] & [T^L_{b1},T^U_{b1}] & [T^L_{13},T^U_{13}]\\ [T^L_{21},T^U_{21}] & [T^L_{b_2},T^U_{b_2}] & [T^L_{23},T^U_{23}]\\ [T^L_{31},T^U_{31}] & [T^L_{b_3},T^U_{b_3}] & [T^L_{33},T^U_{33}]\end{pmatrix}}{det\begin{pmatrix}[T^L_{11},T^U_{11}] & [T^L_{12},T^U_{12}] & [T^L_{13},T^U_{13}]\\ [T^L_{21},T^U_{21}] & [T^L_{22},T^U_{22}] & [T^L_{23},T^U_{23}]\\ [T^L_{31},T^U_{31}] & [T^L_{32},T^U_{32}] & [T^L_{33},T^U_{33}]\end{pmatrix}},$$

$$For\ falsity\ component\ of\ x: x_F = \frac{det\begin{pmatrix}[T^L_{11},T^U_{11}] & [T^L_{12},T^U_{12}] & [T^L_{b1},T^U_{b1}]\\ [T^L_{21},T^U_{21}] & [T^L_{22},T^U_{22}] & [T^L_{b_2},T^U_{b_2}]\\ [T^L_{31},T^U_{31}] & [T^L_{32},T^U_{32}] & [T^L_{b_3},T^U_{b_3}]\end{pmatrix}}{det\begin{pmatrix}[T^L_{11},T^U_{11}] & [T^L_{12},T^U_{12}] & [T^L_{13},T^U_{13}]\\ [T^L_{21},T^U_{21}] & [T^L_{22},T^U_{22}] & [T^L_{23},T^U_{23}]\\ [T^L_{31},T^U_{31}] & [T^L_{32},T^U_{32}] & [T^L_{33},T^U_{33}]\end{pmatrix}}$$

Can be repeat that for y and z and for truth, inderminacy falsity components. Hence, according to Cramer's rule, the Neutrosophic solution is computed component wise as:

$$x = \left(\frac{\left[T^L_{\det(A_N)_x}, T^U_{\det(A_N)_x}\right]}{\left[T^L_{\det_N(A_N)}, T^U_{\det_N(A_N)}\right]}, \frac{\left[I^L_{\det(A_N)_x}, I^U_{\det(A_N)_x}\right]}{\left[I^L_{\det_N(A_N)}, I^U_{\det_N(A_N)}\right]}, \frac{\left[F^L_{\det(A_N)_x}, F^U_{\det(A_N)_x}\right]}{\left[F^L_{\det_N(A_N)}, F^U_{\det_N(A_N)}\right]}\right),$$

$$y = \left(\frac{\left[T^L_{\det(A_N)_y}, T^U_{\det(A_N)_y}\right]}{\left[T^L_{\det_N(A_N)}, T^U_{\det_N(A_N)}\right]}, \frac{\left[I^L_{\det(A_N)_y}, I^U_{\det(A_N)_y}\right]}{\left[I^L_{\det_N(A_N)}, I^U_{\det_N(A_N)}\right]}, \frac{\left[F^L_{\det(A_N)_y}, F^U_{\det(A_N)_y}\right]}{\left[F^L_{\det_N(A_N)}, F^U_{\det_N(A_N)}\right]}\right),$$

$$z = \left(\frac{\left[T^L_{\det(A_N)_z}, T^U_{\det(A_N)_z}\right]}{\left[T^L_{\det_N(A_N)}, T^U_{\det_N(A_N)}\right]}, \frac{\left[I^L_{\det(A_N)_z}, I^U_{\det(A_N)_z}\right]}{\left[I^L_{\det_N(A_N)}, I^U_{\det_N(A_N)}\right]}, \frac{\left[F^L_{\det(A_N)_z}, F^U_{\det(A_N)_z}\right]}{\left[F^L_{\det_N(A_N)}, F^U_{\det_N(A_N)}\right]}\right).$$

Numerical Examples 4.4:

Example 1. Let

$$\begin{cases} ([0.7, 0.9], [0.1,0.2], [0.2,0.1])x \oplus ([0.2,0.4], [0.3,0.5], [0.1, 0.3])y \\ \oplus([0.1,0.3], [0.2,0.4], [0.2,0.6])z = ([0.2, 0.3], [0.0, 0.0], [0.0,0.0]) \\ ([0.3, 0.5], [0.2, 0.4], [0.1,0.3])x \oplus ([0.6, 0.8], [0.1,0.3], [1.0,0.2])y \\ \oplus([0.6, 0.8], [0.1,0.3], [0.1,0.2])z = ([0.0, 0.0], [0.1, 0.3], [0.0,0.0]) \\ ([0.4, 0.6], [0.1, 0.3], [0.2,0.4])x \oplus ([0.3, 0.5], [0.2,0.4], [0.6, 0.8])y \\ \oplus([0.8, 1.0], [0.1,0.2], [0.1,0.3])z = ([0.0,0.0], [0.0, 0.0], [0.4,0.5]) \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ (16)$$

Find solution of the system.

Solution:

Since $\det_N(A_N)$ is defined as:

$$\det_N(A_N) = (T_{\det_N(A_N)}, I_{\det_N(A_N)}, F_{\det_N(A_N)})$$

The main determinant $\det_N(A_N) =$

$$= \left(\begin{vmatrix} [0.7,0.9] & [0.2,0.4] & [0.1,0.3] \\ [0.3,0.5] & [0.6,0.8] & [0.6,0.8] \\ [0.4,0.6] & [0.3,0.5] & [0.8,1.0] \end{vmatrix}, \begin{vmatrix} [0.1,0.2] & [0.3,0.5] & [0.2,0.4] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.1,0.2] \end{vmatrix}, \begin{vmatrix} [0.2,0.1] & [0.1,0.3] & [0.2,0.6] \\ [0.1,0.3] & [1.0,0.2] & [0.1,0.2] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \end{vmatrix} \right)$$

Using Sarrus' rule to calculate the Partial determinants and definition 2.3 we get:

$$T_{det_N(A_N)} = \begin{vmatrix} [0.7,0.9] & [0.2,0.4] & [0.1,0.3] \\ [0.3,0.5] & [0.6,0.8] & [0.2,0.4] \\ [0.4,0.6] & [0.3,0.5] & [0.8,1.0] \end{vmatrix} = [0.7,0.9]det\begin{pmatrix} [0.6,0.8] & [0.2,0.4] \\ [0.3,0.5] & [0.8,1.0] \end{pmatrix} -$$

$$[0.2,0.4]det\begin{pmatrix} [0.3,0.5] & [0.2,0.4] \\ [0.4,0.6] & [0.8,1.0] \end{pmatrix} + [0.1,0.3]det\begin{pmatrix} [0.3,0.5] & [0.6,0.8] \\ [0.4,0.6] & [0.3,0.5] \end{pmatrix}$$

$= [0.7,0.9]([[0.6,0.8][0.8,1.0] - [0.2,0.4][0.3,0.5]) - [0.2,0.4]([0.3,0.5][0.8,1.0] -$

$[0.2,0.4][0.4,0.6]) + [0.1,0.3]([0.3,0.5][0.3,0.5] - [0.6,0.8][0.4,0.6])$

$= [0.7,0.9]([0.48,0.8] - [0.06,0.2]) - [0.2,0.4]([0.24,0.5] - [0.08,0.24]) + [0.1,0.3]$

([0.09,0.25]-[0.0.24,0.48]) $= [0.7,0.9][0.35,0.79] - [0.2,0.4][0.0,0.46] + [0.1,0.3][-0.75,0.13] = [0.25,0.71] - [0.0,0.18] + [-0.075,0.039] = [0.085,0.71] + [-0.075,0.039] = [0.016,0.72]$,

$$I_{det_N(A_N)} = \begin{vmatrix} [0.1,0.2] & [0.3,0.5] & [0.2,0.4] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.1,0.2] \end{vmatrix} = [0.1,0.2]([0.1,0.3][0.1,0.2] - [0.1,0.3][0.2,0.4]) -$$

$[0.3,0.5]([0.2,0.4][0.1,0.2] - [0.1,0.3][0.1,0.3]) + [0.2,0.4]([0.2,0.4][0.2,0.4] - [0.1,0.3][0.1,0.3])$

$= [0.1,0.2]([0.19,0.44] - [0.28,0.58]) - [0.3,0.5]([0.28,0.52] - [0.19,0.51]) + [0.2,0.4]([0.36,0.64] - [0.19,0.51]) = [0.1,0.2][0.0,1.57] - [0.3,0.5][0.55,2.74] + [0.2,0.4][0.71,3.37] = [0.11,1.46] - [0.69,1.87] + [0.77,2.42]$

= [0.059,2.12] + [0.77,2.42] = [0.045,5.13],

$$F_{det_N(A_N)} = \begin{vmatrix} [0.2,0.1] & [0.1,0.3] & [0.2,0.6] \\ [0.1,0.3] & [1.0,0.2] & [0.1,0.2] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \end{vmatrix} = [0.2,0.1]det\begin{vmatrix} [1.0,0.2] & [0.1,0.2] \\ [0.1,0.3] & [0.1,0.3] \end{vmatrix} -$$

$[0.1,0.3]det\begin{vmatrix} [0.1,0.3] & [0.1,0.2] \\ [0.2,0.4] & [0.1,0.3] \end{vmatrix} + [0.2,0.6]det\begin{vmatrix} [0.1,0.3] & [1.0,0.2] \\ [0.2,0.4] & [0.1,0.3] \end{vmatrix} = [0.2,0.1]([1.0,0.2][0.1,0.3] - [0.1,0.2][0.1,0.3])$

$-[0.1,0.3]([0.1,0.3][0.1,0.3] - [0.1,0.2][0.2,0.4]) + [0.2,0.6]([0.1,0.3][0.1,0.3] - [1.0,0.2][0.2,0.4]) = [0.2,0.1]([1.0,0.44] - [0.19,0.44]) - [0.1,0.3]([0.19,0.51]$

$-[0.28,0.52]) + [0.2,0.6]([0.19,0.51] - [1.0,0.52]) = [0.2,0.1][2.27,2.32] - [0.1,0.3][0.37,1.82]] + [0.2,0.6][0.37,0.51] =$ *[2.016,2.19]- [0.43,1.57] + [0. 50,0.80]*

= [1.28,1.39] + [0.50,0.80] = [0.64,1.11].

So, $det_N(A_N) = (T_{det_N(A_N)}, I_{det_N(A_N)}, F_{det_N(A_N)})$

$= ([0.016,0.72],[0.045,5.13],[0.64,1.11])$

determinants for each variable are computed by replacing the corresponding column with the Neutrosophic constants,
For x : $\det_N(A_N)_x = (\det(T_x), \det(I_x), \det(F_x)) =$

$$\left(\begin{vmatrix} [0.2,0.3] & [0.2,0.4] & [0.1,0.3] \\ [0,0] & [0.6,0.8] & [0.6,0.8] \\ [0,0] & [0.3,0.5] & [0.8,1.0] \end{vmatrix}, \begin{vmatrix} [0,0] & [0.3,0.5] & [0.2,0.4] \\ [0.1,0.3] & [0.1,0.3] & [0.1,0.3] \\ [0,0] & [0.2,0.4] & [0.1,0.2] \end{vmatrix}, \begin{vmatrix} [0,0] & [0.1,0.3] & [0.2,0.6] \\ [0,0] & [1.0,0.2] & [0.1,0.2] \\ [0.4,0.5] & [0.6,0.8] & [0.1,0.3] \end{vmatrix}\right)$$

$= ([0.2,0.3]det \begin{vmatrix} [0.6,0.8] & [0.6,0.8] \\ [0.3,0.5] & [0.8,1.0] \end{vmatrix} - [0,0]det \begin{vmatrix} [0.2,0.4] & [0.1,0.3] \\ [0.3,0.5] & [0.8,1.0] \end{vmatrix} +$
$[0,0]det \begin{vmatrix} [0.2,0.4] & [0.1,0.3] \\ [0.6,0.8] & [0.6,0.8] \end{vmatrix}, [0.0,0.0]det \begin{vmatrix} [0.1,0.3] & [0.2,0.4] \\ [0.2,0.4] & [0.1,0.2] \end{vmatrix} - [0.1,0.3]det \begin{vmatrix} [0.3,0.5] & [0.2,0.4] \\ [0.2,0.4] & [0.1,0.2] \end{vmatrix} +$
$[0,0]det \begin{vmatrix} [0.3,0.5] & [0.1,0.3] \\ [0.1,0.3] & [0.1,0.3] \end{vmatrix},$

$[0,0]det \begin{vmatrix} [1.0,0.2] & [0.1,0.2] \\ [0.6,0.8] & [0.1,0.3] \end{vmatrix} - [0,0]det \begin{vmatrix} [0.1,0.3] & [0.2,0.6] \\ [0.6,0.8] & [0.1,0.3] \end{vmatrix} + [0.4,0.5]det \begin{vmatrix} [0.1,0.3] & [0.2,0.6] \\ [1.0,0.2] & [0.1,0.2] \end{vmatrix}) =$
$([0.2,0.3]([0.48,0.8]-[0.18,0.4]) - [0,0]([0.16,0.4]-[0.03,0.15]) + [0.0]([0.12,0.32]-$
$[0.06,0.24]),[0,0]([0.01,0.06]-[0.04,0.16]) - [0.1,0.3]([0.03,0.1]-[0.04,0.16]) + [0,0]([0.03,0.15]-$
$[0.01,0.09]),([0,0]([0.01,0.06]-[0.02,0.12]) - [0,0]([0.01,0.9]-[0.12,0.48]) + [0.4,0.5]([0.01,0.06]-[0.2,0.12])$

$=([0.2,0.3][0.13,0.76],[0,0][0.061,1.5] - [0.1,0.3][0.19,2.5] + [0,0][0.33,15],[0,0][0.83,3] - [0,0][0.021,7.5] +$
$[0.4,0.5][0.083,0.3])$

$= ([0.26,023], [0.061,1.5] - [0.27,2.05] + [0.33,15],[0.83,3] - [0.021,7.5] + [0.45,0.58])$

$= ([0.26,023],[0.030,0.73]] + [0.33,0.15],[0.11,142.9] + [0.45,0.58])$

$= ([0.26,0.23], [0.01,0.11], [0.05,82.88])$

Solving **For x**: using the Neutrosophic division as follows:

$x = det(A_N)_x / det_N(A_N) = (x_T, x_I, x_F)$:

$x = \left(\frac{[0.26,0.23]}{[0.016,0.72]} \quad \frac{[0.01,0.11]}{[0.045,5.13]} \quad \frac{[0.05,82.88]}{[0.64,1.11]}\right) = ([0.36,14.38],[1.24,0.068],[9.64,228.44])$

For $y : det_N(A_N)_y = \left(det(T_y), det(I_y), det(F_y)\right) =$

$$\left(\begin{vmatrix} [0.7,0.9] & [0.2,0.3] & [0.1,0.3] \\ [0.3,0.5] & [0,0] & [0.6,0.8] \\ [0.4,0.6] & [0,0] & [0.8,1.0] \end{vmatrix}, \begin{vmatrix} [0.1,0.2] & [0,0] & [0.2,0.4] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0,0] & [0.1,0.2] \end{vmatrix}, \begin{vmatrix} [0.2,0.1] & [0,0] & [0.2,0.6] \\ [0.1,0.3] & [0,0] & [0.1,0.2] \\ [0.2,0.4] & [0.4,0.5] & [0.1,0.3] \end{vmatrix}\right)$$

$= ([0.7,0.9]det \begin{vmatrix} [0,0] & [0.2,0.4] \\ [0,0] & [0.8,1.0] \end{vmatrix} - [0.2,0.3]det \begin{vmatrix} [0.3,0.5] & [0.6,0.8] \\ [0.4,0.6] & [0.8,1.0] \end{vmatrix}$
$+ [0.1,0.3]det \begin{vmatrix} [0.3,0.5] & [0,0] \\ [0.4,0.6] & [0,0] \end{vmatrix}, [0.1,0.2]det \begin{vmatrix} [0.1,0.3] & [0.1,0.3] \\ [0,0] & [0.2,0.4] \end{vmatrix}$
$- [0,0]det \begin{vmatrix} [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.0,0.2] \end{vmatrix}$
$+ [0.2,0.4]det \begin{vmatrix} [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0,0] \end{vmatrix}, [0.2,0.1] \; det \begin{vmatrix} [0,0] & [0.1,0.2] \\ [0.4,0.6] & [0.1,0.3] \end{vmatrix}$

$-[0,0] \; det \begin{vmatrix} [0.1,0.3] & [0.1,0.2] \\ [0.2,0.4] & [0.1,0.3] \end{vmatrix} + [0.2,0.6] \; det \begin{vmatrix} [0.1,0.3] & [0,0] \\ [0.4,0.6] & [0.4,0.5] \end{vmatrix}$

$= ([0.7, 0.9]\,[0,0] - [0.2,0.3]([0.24,0.5] - [0.24,0.48]) + [0.1,0.3][0,0], [0.1,0.2]([0.28,0.58] - [0.1,0.3]) - [0,0]([0.2,0.52] - [0.19,0.51]) + [0.2,0.4]([0.2, 0.4] - [0.19,0.51]), [0.2,0.1]([0.1,0.3] - [0.46,0.68]) - [0,0]([0.19,0.51] - [0.28,0.52]) + [0.2,0.6]([0.46,0.65] - [0.4, 0.6])$

$= ([0,0] - [0.2,0.3][-0.46, -0.5] + [0,0], [0.1,0.2][0.92,5.8] - [0,0][0.39,2.70] + [0.2,0.4][0.39,1.053], [0.2,0.1][0.15,0.65] - [0,0][0.37,1.82] + [0.2,0.6][0.77,1.63])$

$= ([0.18,0.08], [0.32,0.24] - [0.37,1.82] + [0.08,0.42], [0.32,0.69] - [0.37,1.82] + [0.154,0.98]) = ([0.18,0.08], [0.176,0.65] + [0.08,0.42], [0.176,1.86] + [0.154,0.98]) = ([0.18,0.08], [0.0141, 0.273], [0.0271,1.823])$

Solving For y: using the Neutrosophic division as follows:

$y = det(A_N)_y / det_N(A_N) = (y_T, y_I, y_F)$:

$$y = \left(\frac{[0.18,0.08]}{[0.016,0.72]} \quad \frac{[0.0141, 0.273]}{[0.045,5.13]} \quad \frac{[0.0271,1.823]}{[0.64,1.11]}\right)$$

$= ([0.25,5], \quad [-8.38,0.24], \quad [9.84,3.29])$

For $z : det_N(A_N)_z = (det(T_z), det(I_z), det(F_z))$

$$= \left(\begin{vmatrix} [0.7,0.9] & [0.2,0.4] & [0.2,0.3] \\ [0.3,0.5] & [0.6,0.8] & [0.0,0.0] \\ [0.4,0.6] & [0.3,0.5] & [0.0,0.0] \end{vmatrix}, \begin{vmatrix} [0.1,0.2] & [0.3,0.5] & [0.0,0.0] \\ [0.2,0.4] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.0,0.0] \end{vmatrix}, \begin{vmatrix} [0.2,0.1] & [0.1,0.3] & [0.0,0.0] \\ [0.1,0.3] & [1.0,0.2] & [0.0,0.0] \\ [0.2,0.4] & [0.6,0.8] & [0.4,0.5] \end{vmatrix} \right)$$

$= [0.2, 0.3]([0.3, 0.5][0.3, 0.5] - [0.6, 0.8][0.4, 0.6]) - [0.0, 0.0]([0.7,0.9][0.3, 0.5] - [0.2,0.4][0.4, 0.6]) + [0.0, 0.0]([0.7,0.9][0.6, 0.8] - [0.2,0.4][0.3, 0.5]), [0.0, 0.0]([0.2, 0.4][0.2, 0.4] - [0.1,0.3][0.1,0.3]) - [0.1,0.3]([0.1,0.2][0.2,0.4]$

$-[0.3,0.5][0.1, 0.3]) + [0.0, 0.0]([0.1,0.2][0.1,0.3] - [0.3,0.5][0.2, 0.4]), [0.0, 0.0]([0.1,0.3][0.6, 0.8] - [1.0,0.2][0.2,0.4]) - [0.0, 0.0]([0.2,0.1][0.6, 0.8] - [0.1, 0.3][0.2,0.4]) + [0.4,0.5]([0.2,0.1][1.0,0.2]$

$-[0.1, 0.3][0.1, 0.3]) = [0.2, 0.3]([0.09,0.25] - [0.24,0.48]) - [0.0, 0.0],$

$[0.0, 0.0]([0.36,0.64] - [0.19,0.51]) - [0.1,0.3]([0.28,0.52] - [0.37,0.65]) + [0.0, 0.0]([0.19,0.44] - [0.44,0.7]), [0.0, 0.0]([0.64, 0.86] - [1,0.52]) - [0.0, 0.0]([0.68,0.82] - [0.28,0.58]) + [0.4,0.5]([1,0.28] - [0.19,0.51])$

$$\begin{aligned} &= ([0.2, 0.3][-0.75,0.0132], [0.0, 0.0][0.71,3.37] - [0.1,0.3][0.43,0.8] + [0.0, 0.0][0.27,1], [0.0, 0.0][1.231, 0.86] \\ &\quad - [0.0, 0.0][1.17,1.41] + [0.4,0.5][1.961,1.474]) \\ &= ([-0.15,0.004], [0.71,3.37] - [0.125,0.70] + [0.27,1], [1.231, 0.86] - [1.17,1.41] \\ &\quad + [0.78,0.74]) \end{aligned}$$

$= ([-0.15,0.004], [1.014,26.96] + [0.27,1], [0.87,0.74] + [0.78,0.74])$

$= ([-0.15,0.004], [0.27, 26.96], [0.68,0.55])$

Solving For z: using the Neutrosophic division as follows:

$z = det(A_N)_z / det_N(A_N) = (z_T, z_I, z_F)$:
$z = ([0.016,0.72], [0.045,5.13], [0.64,1.11])$

$$= \left(\frac{[-0.15,0.004]}{[0.016,0.72]}, \quad \frac{[0.27, 26.96]}{[0.045,5.13]}, \quad \frac{[0.68,0.55]}{[0.64,1.11]}\right)$$

$= ([-0.21,0.25], \quad [-0.73,28.18], \quad [3.91, -0.25])$

The fined solution of the system:

$x = ([0.36,14.38], [1.24,0.068], [9.64,228.44])$

$y = ([0.25,5], \quad [-8.38,0.24], \quad [9.84,3.29])$

$z = ([-0.21,0.25], \quad [-0.73,28.18], \quad [3.91,-0.25])$

Mathematical Verification by substitution:

Taking values of the solution $x = (x_T, x_I, x_F)$, $y = (y_T, y_I, y_F)$, $z = (z_T, z_I, z_F)$ and from definition 2.3 we get:

$$([0.7, 0.9], [0.1,0.2], [0.2,0.1])x \oplus ([0.2,0.4], [0.3,0.5], [0.1, 0.3])y \oplus ([0.1,0.3], [0.2,0.4], [0.2,0.6])z$$

Equation 1:
$([0.7, 0.9], [0.1,0.2], [0.2,0.1])x \oplus ([0.2,0.4], [0.3,0.5], [0.1, 0.3])y \oplus ([0.1,0.3], [0.2,0.4], [0.2,0.6])z$

$= ([0.7, 0.9], [0.1,0.2], [0.2,0.1])(x_T, x_I, x_F) \oplus ([0.2,0.4], [0.3,0.5], [0.1, 0.3])(y_T, y_I, y_F)$

$\oplus ([0.1,0.3], [0.2,0.4], [0.2,0.6])(z_T, z_I, z_F)z$

$= ([0.7, 0.9], [0.1,0.2], [0.2,0.1])([0.36,14.38], [1.24,0.068], [9.64,228.44]) \oplus$

$([0.2,0.4], [0.3,0.5], [0.1, 0.3])([0.25,5], \quad [-8.38,0.24], \quad [9.84,3.29]) \oplus$

$([0.1,0.3], [0.2,0.4], [0.2,0.6])([-0.21,0.25], \quad [-0.73,28.18], \quad [3.91,-0.25])$

= ([0.252,12.942], [1.22,0.254],[7.912,205.70])$\oplus$*([0.05,2],[-5.55,0.62],[8.96,2.603])*$\oplus$

$([-0.089,0.48], [-0.384,17.31], [3.33,0.5] =$
$([0.29, -10.942], [-6.77,0.16], [70.89,535.44] \oplus ([-0.089,0.48], [-0.384,17.31], [3.33,0.5]$

$= ([0.23, -6.19], [2.56,2.77], [236.064,267.72]) \neq ([0.2, 0.3], [0.0, 0.0], [0.0,0.0])$

Although a unique mathematical solution exists for the neutrosophic system, a challenge arises when verifying it through direct substitution in the original system. The results showed that full consistency is not achieved upon substitution, reflecting the inherent nature of the vague data processed by neutrosophic logic. In classical systems, we seek an exact solution. Whereas in neutrosophic systems, we aim for an optimal solution under uncertainty.

The discrepancies observed in the substitution process demonstrate the capability of neutrosophic logic to handle uncertainty, as the obtained solutions represent a range of possible values rather than a single value, thereby enhancing flexibility in complex real-world problems.

The lack of substitution consistency in interval neutrosophic systems indicates that the original system is inconsistent and no solution satisfies all intervals simultaneously. The failure to achieve full consistency also reflects the nature of operations in interval neutrosophic systems, where algebraic operations are not entirely invertible as in classical algebra. Hence, neutrosophic logic does not pursue an exact solution but an optimal one under uncertainty. Consequently, neutrosophic systems may have a unique solution; however, this does not guarantee complete consistency upon substitution.

Mathematical analysis of non-identity:

Non-identity of interval operations in classical systems, algebraic operations are completely reversible. However, in neutrosophic interval systems, the operations lose their reversibility.

Loss of information in arithmetic series. Serial operations on intervals lead to a gradual loss of information:

- Loss of interval distribution within intervals.
- Loss of relationships between lower and upper bounds.
- Accumulation of inaccuracy with each operation.

The effect of neutrosophic processes on $([T^L, T^U], [I^L, I^U], [F^L, F^U])$.

Example 2. Let
$$\begin{cases} ([0.6, 0.8], [0.1, 0.3], [0.0,0.2\,])x + ([0.3, 0.5], [0.2, 0.4], [0.1,0.3\,])y \\ +([0.2, 0.4], [0.1, 0.3], [0.0,0.2\,])z = ([0.9, 0.1], [0.0, 0.2], [0.1,0.3\,]) \\ ([0.6, 0.8], [0.1, 0.3], [0.0,0.2\,])x + \big([0.3, 0.5], [0.2,0.4]\big], [0.1,0.3]\big)y \\ +([0.2, 0.4], [0.1, 0.3], [0.0,0.2\,])z = ([0.2, 0.4], [0.1, 0.3], [0.0,0.1\,]) \\ ([0.4, 0.6], [0.2, 0.4], [0.1,0.3\,])x + ([0.2, 0.4], [0.3, 0.5], [0.2,0.4\,])y \\ +([0.3, 0.5], [0.1,0.3], [0.0,0.2\,])z = ([0.8, 1.0], [0.0, 0.2], [0.1,0.3\,]) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots \textit{(17)}$$

Solution: $\det_N(A_N)$

$$= \begin{vmatrix} ([0.6, 0.8], [0.1, 0.3], [0.0,0.2]\,) & ([0.3, 0.5], [0.2, 0.4], [0.1,0.3\,]) & ([0.2, 0.4], [0.1, 0.3], [0.0,0.2\,]) \\ ([0.6, 0.8], [0.1, 0.3], [0.0,0.2\,]) & \big([0.3, 0.5], [0.2,0.4]\big], [0.1,0.3]\big) & ([0.2, 0.4], [0.1, 0.3], [0.0,0.2\,]) \\ ([0.4, 0.6], [0.2, 0.4], [0.1, 0.3]) & ([0.2, 0.4], [0.3, 0.5], [0.2,0.4\,]) & ([0.3, 0.5], [0.1,0.3], [0.0,0.2\,]) \end{vmatrix}$$

$$= \left(\begin{vmatrix} [0.6, 0.8] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.6, 0.8] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.4, 0.6] & [0.2,0.4\,] & [0.3, 0.5] \end{vmatrix}, \begin{vmatrix} [0.1, 0.3] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.1, 0.3] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.2, 0.4] & [0.3, 0.5] & [0.1,0.3\,] \end{vmatrix}, \begin{vmatrix} [0.0,0.2\,] & [0.1, 0.3] & [0.0,0.2\,] \\ [0.0,0.2\,] & [0.1, 0.3] & [0.0,0.2\,] \\ [0.1, 0.3] & [0.2,0.4\,] & [0.0,0.2\,] \end{vmatrix} \right)$$

We notice that in each of the determinants the first row is equal to the second row, so:

$\det_N(A_N) = ([0,0], [0,0]\,, [0,0])$

Now Calculation of Partial determinants for the variables

unknowns $(\det(A_N)_x\,, \det(A_N)_y \det(A_N)_z,)$:

For x : $\det(A_N)_x = (\det(T_x), \det(I_x), \det(F_x))$

$$= \left(\begin{vmatrix} [0.9,0.1\,] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.2, 0.4] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.8,1.0] & [0.2,0.4\,] & [0.3, 0.5] \end{vmatrix}, \begin{vmatrix} [0.0,0.2\,] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.1, 0.3] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.0, 0.2] & [0.3, 0.5] & [0.1,0.3\,] \end{vmatrix}, \begin{vmatrix} [0.1,0.3\,] & [0.1, 0.3] & [0.0,0.2\,] \\ [0.0,0.1\,] & [0.1, 0.3] & [0.0,0.2\,] \\ [0.1, 0.3] & [0.2,0.4\,] & [0.0,0.2\,] \end{vmatrix} \right)$$

Using Sarrus' rule to calculate the Partial determinants and definition 2.3 we get:

$$det\ (T_x) = \begin{vmatrix} [0.9,0.1\,] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.2, 0.4] & [0.3, 0.5] & [0.2,0.4\,] \\ [0.8,1.0] & [0.2,0.4\,] & [0.3, 0.5] \end{vmatrix}$$

$$= [0.9,0.1\,]([0.3, 0.5][0.3, 0.5] - [0.2,0.4\,][0.2,0.4\,]) - [0.3, 0.5]([0.2, 0.4][0.3, 0.5] - [0.2,0.4\,][0.8,1.0]) \\ + [0.2,0.4\,]([0.2, 0.4]([0.2,0.4\,] - [0.3, 0.5][0.8,1.0])$$

$$=[0.9, .1\,]([0.09,0.25] - [0.04,0.16]) - [0.3, 0.5]([0.06,0.2] - [0.16,0.4] + [0.2,0.4\,]([0.04,0.16] - [0.24,0.5])$$

$$= [0.9,0.1\,][-0.08,0.22] - [0.3, 0.5][-0.57,0.048] + [0.2,0.4\,][-0.92,\textit{-0. 12]}$$

$$= [-0.072,0.022\,] - [-0.171, 0.024] + [-0.184, -0.048]$$

$$= [-0.1,0.17] + [-0.184, -0.048] = [-3.00,0.130]] \neq [0,0],$$

$$det(I_x) = \begin{vmatrix} [0.0,0.2\,] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.1, 0.3] & [0.2,0.4\,] & [0.1,0.3\,] \\ [0.0, 0.2] & [0.3, 0.5] & [0.1,0.3\,] \end{vmatrix} = [0.0,0.2\,]([0.2,0.4\,]\,[0.1,0.3\,] - [0.1,0.3\,][0.3, 0.5]) -$$
$$[0.2,0.4\,]([0.1, 0.3][0.1, 0.3] - [0.1, 0.3][0.0, 0.2]) + [0.1,0.3\,]([0.1, 0.3][0.3, 0.5] - [0.2,0.4\,][0.0, 0.2])$$

$$= [0.0,0.2\,]([0.28,0.58] - [0.37,0.65]) - [0.2,0.4\,]([0.19,0.51] - [0.1,0.44]) + [0.1, 0.3]([0.37,0.65] - [0.2,0.52])$$

$$= /[0.0,0.2\,][0.43,1.57] - [0.2,0.4\,][0.43,5.1] + [0.1,0.3\,][0.71,3.25]$$

$=$ *[*0.43,1.46] − [0.54,3.46] + [0.74,2.58] = *[0.124,0.42]* +[0.74,2.58] = *[0.09,1.08]*

$\neq [0,0]$,

$$det\ (F_x) = \begin{vmatrix} [0.1,0.3\ ] & [0.1, 0.3] & [0.0,0.2\ ] \\ [0.0,0.1\ ] & [0.1, 0.3] & [0.0,0.2\ ] \\ [0.1, 0.3] & [0.2,0.4\ ] & [0.0,0.2\ ] \end{vmatrix} = [0.1,0.3\ ]([0.1,0.3\ ][0.0,0.2\ ] - [0.0,0.2\ ][0.2,0.4\ ]) -$$

$[0.1, 0.3]([0.0,0.1\ ][0.0,0.2\ ] - [0.0,0.2\ ][0.1, 0.3]) + [0.0,0.2\ ]([0.0,0.1\ ][0.2,0.4\ ] - [0.1, 0.3][0.1, 0.3]) =$ *(*[0.1,0.3]([0.1,0.44] − [0.2,0.52]) − [0.1, 0.3]*([0.0,0.28]-[0.1,0.44])+*[0.0,0.2]([0.2,0.46] − [0.19,0.51])

$=$ [0.1,0.3]*[0.19,0.85]*–[0.1, 0.3] *[0.0,2.8] +*[0.0,0.2]*[0.39,2.42]*

= [0.27,0.9]- [0.1,2.26] + [0.39, -0.48] = [0.12,9] + [0.39, -0.48] = [0.045,-4.32] ≠ [0,0].

$= [0.1,0.3\ ][-0.08,0.06] - [0.1,0.3\ ][-0.06,0.02] + [0.0,0.2\ ][-0.09,0.03] = [-0.024,0.030] \neq [0,0].$

Since all three components $(T, I, F) =$ (truth component, inderminacy component and falsity component) do not satisfy the solution condition, the neutrosophic system as a whole has no solution. When $\det_N(A_N) = \widetilde{0}, \det(A_j) = 0$ for some j, for any component, the system is inconsistent ([3], [20]). This means that the equations are contradictory and no solution can be found that satisfies all three components.

Example3: Let

$$\begin{cases} ([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])x + ([0.3, 0.5], [0.3, 0.5], [0.1,0.3\ ])y \\ +([0.2, 0.4], [0.1, 0.3], [0.0,0.2\ ])z = ([0.9, 0.1], [0.0, 0.2], [0.1,0.3\ ]) \ldots (i) \\ ([0.2, 0.4], [0.5, 0.7], [0.0,0.6\ ])x + ([0.1, 0.2], [0.7,0.8], [0.5,0.7])y \\ +([0.4, 0.8], [0.5, 0.7], [0.0,0.6\ ])z = ([0.5, 0.03], [0.0, 0.6], [0.5,0.7\ ]) \ldots (ii) \\ ([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])x + ([0.3, 0.5], [0.2, 0.4], [0.1,0.3\ ])y \\ +([0.2, 0.4], [0.1,0.3], [0.0,0.2\ ])z = ([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) \ldots (iii) \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Find solution of the system.

Solution: we note that from the system (18) equation (i) and equation (iii) are exactly identical. And equation (ii) is a multiple of equation (i). So the system reduces to a single independent equation:

$([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])x + ([0.3, 0.5], [0.2, 0.4], [0.1,0.3\ ])y + ([0.2, 0.4], [0.1,0.3], [0.0,0.2\ ])z = ([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) \ldots .$ (v)

Therefor general solution (choosing z as a free variable):

z = t = ([t, t], [t, t], [t, t]) (any value within [0, 1]. By substitute the value of z into equation (4), we find that:

$$([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])x + ([0.3, 0.5], [0.2, 0.4], [0.1,0.3\ ])y =$$

$$([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) - ([0.2, 0.4], [0.1,0.3], [0.0,0.2\ ])([t, t], [t, t], [t, t])$$

Choose y = ([0,0], [0,0] , [0,0]) we get:

$$x = \frac{([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) - ([0.2, 0.4], [0.1,0.3], [0.0,0.2\ ])([t, t], [t, t], [t, t])}{([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])}$$

$$= \frac{([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) - ([0.2t, 0.4t], [0.1 + 0.9t, 0.3 + 0.7], [t, 0.2 + 0.8t\ ])}{([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])}$$

$$= \frac{([09, 0.1], [0.0, 0.2], [0.1,0.3\ ]) - ([0.2t, 0.4t], [0.1 + 0.9t, 0.3 + 0.7], [t, 0.2 + 0.8t\ ])}{([0.6, 0.8], [0.1, 0.3], [0.0,0.2\ ])}$$

$$S = \{(x,y,z) \in R^3 \cup \{I\}: x = \{\frac{([09,0.1],[0.0,0.2],[0.1,0.3\,]) - ([0.2t,0.4t],[0.1+0.9t,0.3+0.7],[t,0.2+0.8t\,])}{([0.6,0.8],[0.1,0.3],[0.0,0.2\,])},$$
$$y = ([0,0],[0,0],[0,0]),\;\; z = ([t,t],[t,t],[t,t])\}.$$

So the ternary (3×3) neutrosophic systemhas an infinite number of solutions

(**one free variable**).

Conclusion:

In conclusion, this study has successfully addressed the challenge of solving tri-component interval-valued neutrosophic linear systems by developing a novel and generalized formulation of Cramer's rule. The proposed methodology framework extends the classical tenets of linear algebra into the domain of neutrosophic logic, providing a robust analytical tool for handling the inherent uncertainty prevalent in complex real-world data. The primary theoretical contribution of this work lies in the derivation of a generalized matrix form and the subsequent axiomatic development of the solution method, which has been rigorously validated through comprehensive numerical examples.

These examples not only demonstrate the method's efficacy in obtaining unique solutions but also its capability to conclusively identify cases of no solution or infinitely many solutions within a neutrosophic environment. The implications of this research are twofold: theoretically, it enriches the field of neutrosophic algebra and opens new avenues for extending other classical solution theorems; practically, it offers a powerful computational paradigm for experts and decision-makers in fields such as engineering, economics, and artificial intelligence, where modeling under ambiguount. Despite its promising results, this work presents several avenues for future research. Potential directions include the exploration of algorithmic implementations for large-scale systems, integration with optimization techniques, and application to real-case studies in predictive modeling and multi-criteria decision analysis. Ultimately, this study establishes a foundational step towards more sophisticated and reliable uncertainty-based modeling, underscoring the transformative potential of neutrosophic logic in advancing computational intelligence.

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