A Graphic Presentation of Some Bitopological Spaces

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Abstract:

Given a bitopological space (X,τ_1,τ_2), where both (X,τ_1) and (X,τ_2) belong to a certain class of topological spaces , we will show that there exist a graph $G=(X,\sigma_1,\sigma_2)$ which will give a graphic presentation of the bitopological space (X,τ_1,τ_2) .

Keywords: Graph; Bitopology; maps; Idempotent.

1. Preliminaries:

1-1. Definition: If X is a set, a map $\sigma : X \to X$ is said to be an idempotent map if $\sigma \circ \sigma = \sigma$.

1-2. Lemma: If $\sigma : X \to X$ is any idempotent map, $C_\sigma : P(X) \to P(X)$ defined by $C_\sigma(A) = A \cup \sigma(A)$ for any $A \in P(A)$, then C_σ is a closure operation in the set X .

Proof: see [1].

1-3. Definition: If X is a non-empty set, $\sigma : X \to X$ is an idempotent map. Let τ_σ denotes the topology on X such that: $\overline{A} = A \cup \sigma(A)$ for any $A \subseteq X$. We call τ_σ the topology induced by the idempotent map σ .

1-4. Definition: A space X is said to be $T_{1/2}$ space if and only if each one-point set is either open or closed in X .

The following theorem is 1.5.6 of [1].

1-5. Theorem: If τ_σ is the topology induced by an idempotent map $\sigma : X \to X$, then the frontier of any one-point set is either empty or a one-point set.

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1-6. Theorem: If τ_σ is the topology induced by an idempotent map $\sigma : X \to X$, then (X, τ_σ) is a $T_{1/2}$ space.

Proof:

Let $x \in X$, since $\sigma : X \to X$ is an idempotent map then either we have $\sigma(x) = x$ or $\sigma(x) = y$, $x \neq y$, and $\sigma(y) = y$.

If $\sigma(x) = x$, then $\overline{\{x\}} = \{x\} \cup \sigma(\{x\}) = \{x\}$. So $\{x\}$ is a closed set.

If $\sigma(x) = y$ where $x \neq y$, $\sigma(y) = y$. Then $\overline{\{x\}} = \{x\} \cup \sigma(\{x\}) = \{x, y\} = \{x\} \cup Fr\{x\}$, and so by 1-5 $Fr\{x\} = \{y\}$.

Since $\{x\}^{\circ} = \{x\} \setminus Fr\{x\} = \{x\} \setminus \{y\} = \{x\}$, so $\{x\}$ is an open set.

1-7. Definition: A graph G is a triple (V, E, ψ), where V is a non-empty set called the set of vertices, E is a set disjoint from V called the set of edges, and ψ is a map from E into $V \times V$ called the incident map.

A graph $G = (V, E, \psi)$ is said to be directed graph if each edge is associated with an ordered pair of $V \times V$.

Now let $G = (V, E, \psi)$ be a directed graph, $\pi_i : V \times V \to V$ be the projection maps for $i = 1,2$, and let $d_i = \pi_i \circ \psi$ for $i = 1,2$. If we put $X = V \cup E$ and we let $\sigma_i : X \to X$ be the map defined by:

$$\sigma_i(x) = \begin{cases} x & if \quad x \in V \\ d_i(x) & if \quad x \in E \end{cases}$$

for $i = 1,2$. Then σ_i is an idempotent map for, and σ_1, σ_2 satisfy the following composition property:

$\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_1 = \sigma_1$,and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_2 = \sigma_2$

So following [4] we can formalize the following equivalent definition of a directed graph.

1-8. Definition: A directed graph G is a triple (X, σ_1, σ_2), where X is a non-empty set and σ_1, σ_2 are two unary operations on X satisfying the following composition property $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_1 = \sigma_1$, and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_2 = \sigma_2$.

1-9. Definition: If X is a set and τ_1,τ_2 are two topologies on X, then the triple (X,τ_1,τ_2) is called a bitopological space.
Notice that if (X,τ_1,τ_2) is a bitopological space, $A \subseteq X$ then A is said to be τ_i-open if $A \in \tau_i$ for $i = 1,2$. And we say that (X,τ_1,τ_2) is a $T_{1/2}$ bitopological space if both $(X,\tau_1),(X,\tau_2)$ are $T_{1/2}$ spaces.

2. Graphic presentation:

In [4] Waldemar Korczynski gave a topological presentation of a graph and in [1] a topological presentation of a directed graph was given. In the following theorem, we will prove that the other way around works for a certain class of bitopological spaces.

2-1. Theorem: If (X,τ_1,τ_2) is a $T_{1/2}$ bitopological space, $Fr_{\tau_i}\{x\}$ is a one-element set or empty for all $x \in X$ and all i, and $\{x\}$ is τ_1-closed if and only if $\{x\}$ is τ_2-closed for all $x \in X$. Then there exist a graph $G = (X,\sigma_1,\sigma_2)$ presenting the bitopological space (X,τ_1,τ_2).

Proof:

Let $\sigma_i : X \to X$ defined by:

$$\sigma_i(x) = \begin{cases} y & \text{if } \ Fr_{\tau_i}\{x\} = \{y\}, \quad y \neq x \\ x & \text{if } \ Fr_{\tau_i}\{x\} = \{x\} \text{ or } \ Fr_{\tau_i}\{x\} = \phi \end{cases}$$

Then σ_i is an idempotent map for all i and $\sigma_2 \circ \sigma_1 = \sigma_1$, $\sigma_1 \circ \sigma_2 = \sigma_2$. For let $x \in X$.

Case (1) If $\{x\}$ is closed. Then $Fr_{\tau_i}\{x\} = \{x\}$ or $Fr_{\tau_i}\{x\} = \phi$. So $(\sigma_i \circ \sigma_i)(x) = \sigma_i(\sigma_i(x)) = \sigma_i(x)$.

Also $(\sigma_2 \circ \sigma_1)(x) = \sigma_2(\sigma_1(x)) = \sigma_2(x) = x = \sigma_1(x)$, and $(\sigma_1 \circ \sigma_2)(x) = \sigma_1(\sigma_2(x)) = \sigma_1(x) = x = \sigma_2(x)$.

Case (2) If $\{x\}$ is open. Then $Fr_{\tau_1}\{x\} = \{y\}, y \neq x$, and $Fr_{\tau_2}\{x\} = \{z\}, z \neq x$.

If $\{y\},\{z\}$ are t_i-closed. Then since $Fr_{t_i}(\{t\}) = Fr_{t_i}[Fr_{t_i}(\{x\})] \subseteq Fr_{t_i}(\{x\}) = \{t\}$, where $t = y$ or z; that is $Fr_{t_1}(\{y\}) \subseteq \{y\}$, $Fr_{t_2}(\{z\}) \subseteq \{z\}$. So $Fr_{t_1}(\{y\}) = \{y\}$ or $Fr_{t_1}(\{y\}) = f$, and $Fr_{t_2}(\{z\}) = \{z\}$ or $Fr_{t_2}(\{z\}) = f$. Hence $(s_i \circ s_i)(x) = s_i(x)$ for $i = 1,2$. Also $(s_2 \circ s_1)(x) = s_1(x)$, and $(s_1 \circ s_2)(x) = z = s_2(x)$.

The case $\{y\}$ is t_1-open or $\{z\}$ is t_2-open is impossible. Because without loss of generality if $\{y\}$ is t_1-open and since $\{x\}$ is t_1-open, then $f = Int_{t_1}(Fr_{t_1}(\{x\})) = Int_{t_1}(\{y\}) = \{y\}$ a contradiction.

Therefore s_1, s_2 satisfy the composition property:

$s_2 \circ s_1 = s_1 \circ s_1 = s_1$, and $s_1 \circ s_2 = s_2 \circ s_2 = s_2$ And hence the triple (X, s_1, s_2) is a graphic presentation of the bitopological space (X, t_1, t_2).

2-2 Example:

Let (X, t_1, t_2) be the bitopological space where $X = \{a, b, c, e_1, e_2\}$ and,

$t_1 = \{f, X, \{b\}, \{e_1\}, \{e_2\}, \{b, e_1\}, \{b, e_2\}, \{e_1, e_2\}, \{a, e_1\}, \{c, e_2\}, \{a, b, e_1\}, \{a, e_1, e_2\}, \{b, c, e_2\}, \{b, e_1, e_2\}, \{c, e_1, e_2\}, \{a, b, e_1, e_2\}, \{a, c, e_1, e_2\}, \{b, c, e_1, e_2\}\}$, and

$t_2 = \{f, X, \{a\}, \{e_1\}, \{e_2\}, \{a, e_1\}, \{a, e_2\}, \{e_1, e_2\}, \{b, e_2\}, \{c, e_1\}, \{a, e_1, e_2\}, \{b, e_1, e_2\}, \{c, e_1, e_2\}, \{a, b, e_2\}, \{a, c, e_1\}, \{a, b, e_1, e_2\}, \{a, c, e_1, e_2\}, \{b, c, e_1, e_2\}\}$. Then

$Fr_{t_1}(\{a\}) = \{a\}$, $Fr_{t_1}(\{b\}) = \{b\}$, $Fr_{t_1}(\{c\}) = \{c\}$, $Fr_{t_1}(\{e_1\}) = \{a\}$, and $Fr_{t_1}(\{e_2\}) = \{c\}$.

Let $s_1 : X \rightarrow X$ be the map defined by :

$$s_1(x) = \begin{cases} x & \text{if} \quad x \neq e_1 \text{ and } x \neq e_2 \\ a & \text{if} \quad x = e_1 \\ c & \text{if} \quad x = e_2 \end{cases}$$

And $Fr_{t_2}(\{a\}) = \{a\}$, $Fr_{t_2}(\{b\}) = \{b\}$, $Fr_{t_2}(\{c\}) = \{c\}$, $Fr_{t_2}(\{e_1\}) = \{c\}$, and $Fr_{t_2}(\{e_2\}) = \{b\}$.

Let $s_2 : X \to X$ be the map defined by :

$$s_2(x) = \begin{cases} x & \text{if} \quad x \neq e_1 \text{ and } x \neq e_2 \\ c & \text{if} \quad x = e_1 \\ b & \text{if} \quad x = e_2 \end{cases}$$

Then figure 1.1 is the directed graph (X, s_1, s_2) which presents the bitopological space (X, t_1, t_2)

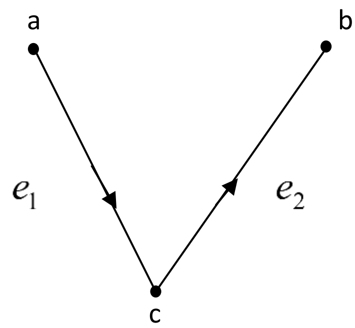

Figure 1.1

عرض بياني لبعض الفضاءات التبولوجية الثنائية

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المستخلص:

إذا أعطي أي فضاء توبولوجي ثنائي (X,t_1,t_2) حيث كل من (X,t_1) ، (X,t_2) ينتمي إلى صنف محدد من الفضاءات التوبولوجية. في هذه الورقة سوف نثبت بأنه يوجد بيان موجه $G=(X,s_1,s_2)$ يمثل عرض للفضاء التوبولوجي الثنائي (X,t_1,t_2).

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