Commutativity of prime near _ rings

Ali M. Sager *

Abstract:

In this paper, we examine Golbasi's approach towards the commutativity in prime near-rings with generalized(σ,τ)-derivations, showing inconsistency in his proof due to its dependence on the implicit assumption of left and right distributivity. To make this proof correct and complete, we provide a more rigorous alternative proof based on a suitable extension of lemma given in [4].

Introduction:

Over the last few decades a lot of work has been done on commutativity of prime rings (see for example [9]); it is natural to look for comparable results on near-rings. A near-ring is a set N together with two binary operations $+, \cdot$ such that

(1). $(N,+)$ is a group (not necessarily abelian).

(2). $(N, \cdot)$ is a semi group.

(3). $x \cdot (y+z) = x \cdot y + x \cdot y \quad \forall x, y, z \in N$.

To be more precise, this is a left near- ring because satisfied only the left distributive. A near-ring N is said to be zero- symmetric if $0 \cdot x = 0 \quad \forall x \in N$. Throughout this paper N will denote a zero-symmetric left near- ring with multiplicative center Z .An additive endomorphism d of a near-ring N is called derivation on N if

* Department of Mathematics, Faculty of Science, University of Tripoli, Libya

$d(xy) = d(x)y + xd(y)\ \forall x, y \in N$; elements x of N for which $d(x) = 0$ are called constants. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. N will be called prime near-ring if $a, b \in N$ and $aNb = \{0\}$ imply that $a = 0$ or $b = 0$. Historically speaking the study of derivation of near-rings was initiated by H. E. Bell and G. Mason in 1987 [3]. An analogue of Posner result in prime near-rings was obtained by Beidar in [2]. On the other hand some results concerning commutativity in prime near-rings with derivation have been generalized in several ways. The main goal of this workis result due to Golbasireads as follows: Let N be a prime near-ring with nonzero generalized (σ, τ) - derivation f associated with d such that $\tau f = f\tau$, $\sigma f = f\sigma$. If $[f(N), f(N)] = 0$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.Here we should mention that the proof given in [6] was not correct. (At one point both left and right distributivity were assumed). Our present paperis to give an alternative proof of this result. To achieve this we need first to impose further restrictions on the derivation$d$,we also need to extend Lemma given in [4, lemma 2.2] to become applicable in our case, where this lemma establish the following, if $d(f(N)) = 0$, then $f(d(N)) = 0$.The arrangement of the paper is as follows. Notations and some preliminary facts will be given in section 2. The proof of our main result will be given in section 3. The conclusion will be given in section 4.

Preliminaries:

Lemma 2.1.[1] Let d be an arbitrary derivation on the near- ring N. Then N satisfies the following partial distributive law:

(1) $[xd(y) + d(x)y]z = xd(y)z + d(x)yz\ \forall x, y, z \in N$;

$(2)[d(x)y + xd(y)]z = d(x)yz + xd(y)z \quad \forall x, y, z \in N.$

$\Pr oof\,(2): d[(xy)z] = d(xy)z + xyd(z) = [d(x)y + xd(y)]z + xyd(z).$ *And*

$$d[x(yz)] = d(x)yz + xd(yz)$$
$$= d(x)yz + xd(y)z + xyd(z).$$

Comparingthetwo expression we obtain

$$[d(x)y + xd(y)]z = d(x)yz + xd(y)z.$$

Definition 2.2. Let N be a near-ring and d a derivation of N. An additive mapping $f: N \to N$ is said to be right generalized derivation of N associated with d if $f(xy) = f(x)y + xd(y)$ for all $x, y \in N$, and f is said to be left generalized derivation of N associated with d if $f(xy) = d(x)y + xf(y)$ for all $x, y \in N$. Finally, f is said to be generalized derivation if it is both left and right generalized derivation of N associated with d.

Definition2.3. Let σ, τ be two automorphisms of a near-ring N. An additive mapping $d: N \to N$ iscalled (σ, τ) -derivationif $d(xy) = \sigma(x)d(y) + d(x)\tau(y), \forall x, y \in N$.

Lemma2.4[1].Let d be (σ,τ)-derivation of N .Then $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$.

Definition 2.5[6] Let N be near-ring, d derivation of N. An additive mapping $f: N \to N$ is said to be left generalized (σ, τ)-derivation of N associated with d if $f(xy) = d(x)\tau(y) + \sigma(x)f(y)$, and f is said to be right generalized (σ, τ) -derivation of N associatedwith d if $f(xy) = f(x)\tau(y) + \sigma(x)d(y), \forall x, y \in N$. f is said to be (σ, τ)-derivation of N associated with d, if it is both left and right generalized (σ, τ)-derivation of N associated with d.

Lemma2.6[6] (i)-Let f be right generalized $(\sigma,\tau)-$derivation of a near-ring N associated with d. Then
$f(xy)=\sigma(x)d(y)+f(x)\tau(y),\ \forall x,y,\in N.$
(ii)- Let f be a left generalized (σ,τ)-derivation of near-ring N associated with d. Then $f(xy)=\sigma(x)f(y)+d(x)\tau(y),\ \forall x,y\in N.$
Lemma 2.7[6] (i)-Let f be a right generalized (σ,τ)-derivation of near-ring N associated with d .Then
$[f(x)\tau(y)+\sigma(x)d(y)]\tau(z)= f(x)\tau(y)\tau(z)+\sigma(x)d(y)\tau(z)$
(ii)-Let f be a left generalized (σ,τ)-derivation of near-ring N withassociated d. Then
$[d(x)\tau(y)+\sigma(x)f(y)]\tau(z)=d(x)\tau(y)\tau(z)+\sigma(x)f(y)\tau(z),\ \forall x,y,z\in N.$
Lemma 2.8[6]- Let N be a prime near-ring, f a generalized (σ,τ) -derivation of N associated with d and $a\in N$. Then the following hold: (i) - If $af(N)=0,$ then $a=0.$
(ii) - If $f(N)\tau(a)=0,$ then $a=0.$
Lemma 2.9[5]- Let 1 N be a prime near-ring, d a nonzero (σ,τ)-derivation of N with associated d and $a\in N$.
Then the followinghold:
(1)- If $d(N)\sigma(a)=0,$ then $a=0$.
(2)- If $ad(N)=0,$ then $a=0$.
Proof (1): For any $x,y\in N,$ we have
$0=d(xy)\sigma(a)=[\sigma(x)d(y)+d(x)\tau(y)]\sigma(a)=\sigma(x)d(y)\sigma(a)+d(x)\tau(y)\sigma(a).$
Using hypothesis, we get $d(x)N\sigma(a)=0,$ since N is prime and $d\neq 0\ \Rightarrow a=0.$
Theorem 2.10.[6]- Let N be a prime near-ring with a nonzero (σ,τ) -derivation f associated with d. If $f(N)\subset Z,$ then $(N,+)$ is abelian. Moreover, if N is 2- torsion free, then N is commutative ring.

Main Results:

Theorem 3.1. Let N be a prime near-ring with a nonzero generalized (σ,τ)-derivation f associated with nonzero $(\sigma,\tau)-$derivation d such that $f\sigma=\sigma f, \tau f=f\tau$ and $d\sigma=\sigma d, \tau d=d\tau$. If $[f(N),f(N)]=0$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is commutative ring.

In order to establish the proof of this result we need the following lemma (which is a generalization of lemma given in [4, lemma 2.2]).

L**emma 3.2:-** Let N be a prime near-ring, and f be generalized (σ,τ)-derivation associated with nonzero $(\sigma,\tau)-$derivation d such that $f\sigma=\sigma f, \tau f=f\tau$ and $d\sigma=\sigma d, \tau d=d\tau$. If $d(f(N))=0$, then $f(d(N))=0$.

Proof. We are assuming that $d(f(x))=0$, for all $x\in N$. It follows that

$$0=d(f(xy))=d[f(x)\tau(y)]+d[\sigma(x)d(y)]$$
$$=\sigma(f(x))d(\tau(y))+\sigma^2(x)d^2(y)+d(\sigma(x))\tau(d(y)) \qquad (1)$$

Appling d to (1) we obtain

$0=d[\sigma(f(x))d(\tau(y))]+d[\sigma^2(x)d^2(y)]+d[d(\sigma(x))\tau(d(y))]$ so we get

$$0=\sigma^2\big(f(x)\big)d^2\big(\tau(y)\big)+\sigma^3(x)d^3(y)+d\big(\sigma^2(x)\big)\tau\big(d^2(y)\big)+ d(\sigma^2(x))\tau(d^2(y))+d^2(\sigma(x))\tau^2(d(y)) \qquad (2)$$

Replacing y by $d(y)$ in (1) gives

$$0=\sigma(f(x))d^2(\tau(y))+\sigma^2(x)d^3(y)+d(\sigma(x))\tau(d^2(y)) \qquad (3)$$

Replacing x by $\sigma(x)$ in (3), we get, `

$$0=\sigma^2(f(x))d^2(\tau(y))+\sigma^3(x)d^3(y)+d(\sigma^2(x))\tau(d^2(y)) \qquad (4)$$

Hence (2) yields

$$0=d\big(\sigma^2(x)\big)\tau\big(d^2(y)\big)+d^2\big(\sigma(x)\big)\tau^2\big(d(y)\big) \quad (5)$$

Replacing x by $d(x)$ in (1) we obtain

$$0=\sigma(f(d(x))d(\tau(y))+\sigma^2(d(x))d^2(y)+d^2(\sigma(x))\tau(d(y)) \qquad (6)$$

Taking $\tau(y)$ for y in (6) obtaining

$$0=\sigma(f(d(x))d(\tau^2(y))+\sigma^2(d(x))d^2(\tau(y))+d^2(\sigma(x))\tau^2(d(y)) \qquad (7)$$

Using (5) we obtain

$$\sigma(f(d(x))d(\tau^2(y))=0 \qquad (8)$$

Replacing x by $\sigma^{-1}(x)$ in (8), we get

$$f(d(x))d(\tau^2(y))=0 \quad \forall x,y,\in N$$

Thus by (2.9)(ii) $f(d(x))=0 \ \forall x\in N$. And this completes the proof of the lemma.

The preceding lemma will now be used to establish the correct proof of the main theorem.

Proof of Theorem 3.1 **.**If z and $z+z$ commute element wise with $f(N)$, then for any $x,\ y\in N$,

$$(z+z)(f(x)+f(y))=(f(x)+f(y))(z+z)$$

Soweget $zf(x)+zf(y)-zf(x)-zf(y)=0$ and thisgives

$$zf(x,y)=0 \quad \forall\ x,\ y\in N \qquad (9)$$

Substituting $f(t),\ t\in N$ for z in (9) we get, $f(t)f(x,y)=0$, since τ is an automorphism of N, we have $\tau(f(t))\tau(f(x,y))=0$, using $\tau f=f\tau$ we infer $f(\tau(t))\tau(f(x,y))=0\ \forall x,y,t\in N$. By lemma (2.8)(ii), we get $f(x,y)=0\ \forall x,y,\in N$. Now, for $w\in N$, we have $0=f(wx,wy)=f(w(x,y))=d(w)\tau(x,y)+\sigma(w)f(x,y)$, and this gives $d(w)\tau(x,y)=0$.

Replacing w by wr and using lemma (2.7)we obtain.

$d(w)\tau(r)\tau(x,y)+\sigma(w)d(r)\tau(x,y)=0$, and this gives $d(w)N\tau(x,y)=0\ \forall x,y,w\in N$. Since N is prime near-ring , $d\neq 0$, we get $(x,y)=0$, so $x+y-x-y=0$, and this gives $x+y=y+x$ hence $(N,+)$ is abelian.

Now, assume that N is 2- torsion free. In view of our hypothesis we have $f(\tau(z))f(f(x)y)=f(f(x)y)f(\tau(z))\ \forall x,y,z\in N$. Using $\tau f=f\tau,\ \sigma f=f\sigma$ and lemma (2.7) we get

$f(\tau(z))d(f(x))\tau(y)+f(\tau(z))\sigma(f(x))f(y)=d(f(x))\tau(y)\tau(f(z))+\sigma(f(x))f(y)\tau(f(z)),$

and this gives $f(\tau(z))d(f(x))\tau(y)=d(f(x))\tau(y)\tau(f(z))$ (10)

If we take yw instead of y in (10), then

$f(\tau(z))d(f(x))\tau(y)\tau(w)=d(f(x))\tau(y)\tau(w)\tau(f(z)),$ so we have

$d(f(x))\tau(y)f(\tau(z))\tau(w)=d(f(x))\tau(y)\tau(w)f(\tau(z))\quad \forall x,y,z,w\in N$,

hence $d(f(x))N[f(\tau(z)),\tau(w)]=0,$ since N is prime we have $d(f(x))=0$ or $f(N)\subseteq Z.$

Assume that $d(f(x))=0$ then

$0=d(f(xy))=d[d(x)\tau(y)+\sigma(x)f(y)]=d(d(x)\tau(y))+d(\sigma(x)f(y)),$ so

we have $d^2(x)\tau^2(y)+\sigma(d(x))d(\tau(y))+d(\sigma(x))\tau(f(y))=0\quad \forall x,y,\in N.$

Replacing y by $\tau^{-1}(y)$ in the last equation, we get

$d^2(x)\tau(y)+\sigma(d(x))d(y)+d(\sigma(x))f(y)=0\quad \forall x,y,\in N$ (11).

Our claim is to show that $f(d(x)y)=0\quad \forall x,y\in N.$

First we notice that

$f(d(x)y)=d^2(x)\tau(y)+\sigma(d(x))f(y)$ (12)

$f(d(x)y)=f(d(x))\tau(y)+\sigma(d(x))d(y),$ but by last lemma

$f(d(x))=0,$ so we get

$f(d(x)y)=\sigma(d(x))d(y)$ (13)

Substituting (12) and (13) in (11), we get $2f(d(x)y)=0$, but N is 2-torsion, so $f(d(x)y)=0$,and this proves the claim.

Now, writingyzinstead of y in (11) we get

$d^2(x)\tau(yz)+\sigma(d(x))d(yz)+d(\sigma(x))f(yz)=0.$

And from this weobtain.

$d^2(x)\tau(yz)+\sigma(d(x))d(y)\tau(z)+\sigma(d((x))\sigma(y)d(z)+$
$d(\sigma(x))f(y)\tau(z)+d(\sigma(x))\sigma(y)d(z)=0.$

By rearranging and using lemma (2.7), we obtain

$[d^2(x)\tau(y)+d(\sigma(x))f(y)]\tau(z)+\sigma(d(x))d(y)\tau(z)+2\sigma(d(x))\sigma(y)d(z)=0$ (15)

From (12), (13), (15) and using the last claim we get,

$2\sigma(d(x))\sigma(y)d(z)=0 \quad \forall\ x,y,z \in N.$ Since N is 2- torsion free, we get, $d(N)\,N\,d(N)=0,$ but N is prime near- ring, so
$d(N)=0 \ \Rightarrow\ d=0$ which is a contradiction, thus we have
$f(N)\subseteq Z.$ Hence by Theorem (2.10) N is a commutative ring. This completes the proof of the theorem.

Conclusion:

Our work is based essentially on work of Golbasi in [6]. However, a careful examination of the proof of theorem 9 in [6] shows that in some stages is not correct (At one point left and right distributivity were assumed). Hence forth we are interested in producing alternative proof to this result. In order to achieve this we need to impose further restriction on the derivation d.We also need to extend lemma 2.2 in [4] to generalized (σ,τ)-derivation in prime near-rings, this version will now be utilized in order to give correct proof of the main result of this paper. Up to this point, we have developed adequate machinery that will enable us to establish the correct proof.

التبديل في قرب الحلقات الأولية

علي محمد صقر *

المستخلص:

في هذه الورقة ندقق في طريقة قولباسي (Golbasi) لدراسة التبديل في قرب الحلقات الأولية مع تعميم الاشتقاق، حيث لوحظ أن الباحث في المرجع [6] في برهان أحد نتائجه الرئيسة افترض وجود التوزيع من اليمين واليسار وهذا غير صحيح، ولهذا فالهدف الأساسي لهذا العمل هو تصحيح هذا البرهان، ولكي يتم ذلك نحتاج إلى توسيع وتطوير مبرهنة مساعدة موجودة في المرجع [4].

* قسم الرياضيات –كلية العلوم – جامعة طرابلس

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